18.369 Midterm Exam (Spring 2018)

You have two hours. The problems have equal weight, so divide your time accordingly.

Problem 1: Irreps

As shown in figure 1, we arrange N identical masses $m > 0$ onto a circle, uniformly spaced, and attach each to its neighbors by a spring constant $\kappa > 0$. The masses are constrained to move along the circle, and the motion of each mass is described by an angle ϕ_ℓ as shown, where $\phi_\ell = 0$ corresponds to the initial position for mass ℓ.

If we assume a time-dependence $e^{-i\omega t}$ as usual, then the frequencies ω satisfy the eigenproblem $\hat{\Theta}\psi = \omega^2\psi$, where $\psi = (\phi_1, \phi_2, \cdots, \phi_N)^T$ and $\hat{\Theta}$ is the $N \times N$ real-symmetric positive-semi-definite matrix:

$$\hat{\Theta} = \frac{\kappa}{m}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Obviously, the system in figure 1 is invariant under C_N rotations, corresponding to a *cyclic shift* $\phi_1 \to \phi_2$, $\phi_2 \to \phi_3$, ..., $\phi_{N-1} \to \phi_N$, $\phi_N \to \phi_1$.

(a) Let $D(n)$ be the representation matrix for a rotation C_N^n (i.e. a cyclic shift n times). **What are the possible irreducible representations** for this group (the *cyclic group* of order N)? [Hint: $D(n)D(n') = D(?)$.] Be sure to get the **right number** of irreps!

(b) Using your answer from (a), **solve for the eigenfrequencies ω and the corresponding eigenvectors**.

(c) This structure *also* has mirror symmetries σ. If N is an *odd* number, then:

  (i) How many mirror symmetry planes are there?

  (ii) What are the conjugacy classes of the symmetry if you include *both* the translations *and* the mirror planes?

  (iii) How many irreps are there, and what are their dimensions? Does this match the degeneracies of your eigenvalues in (b)?

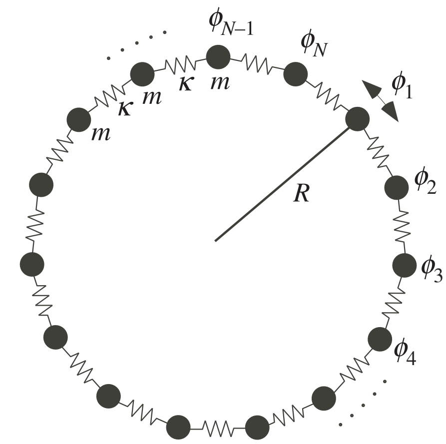

Figure 1: N identical masses m arranged on a circle, connected with spring constants κ, and allowed to slide freely on the circle, where ϕ_ℓ denotes the angular displacement of the ℓ-th mass from its initial position (equally spaced).

Problem 2: Index guiding

In class and in homework, you considered the problem of index-guiding: localization in a higher-index region with translational symmetry. In this problem, you should do the same thing but with a different wave equation, the Schrödinger equation, whose eigenmode equation for time-harmonic modes $\psi(\mathbf{x})e^{-i\omega t}$ is:

$$\hat{H}\psi = \underbrace{\left(-\nabla^2 + V\right)}_{\hat{H}}\psi = \omega\psi$$

where $V(\mathbf{x})$ is a "potential" function. In particular, we consider an x-independent potential $V(y)$ in 2d, as depicted in figure 2, that $= 0$ for $|y| > h$ and is otherwise negative "on average," i.e. $\int_{-\infty}^{\infty} V(y)\,dy < 0$. You are also given that $\int_{-\infty}^{\infty} |V|\,dy$ is finite.

Note that $\hat{H}$ is Hermitian under the usual inner product $\langle\phi, \psi\rangle = \int \phi^*\psi$ for functions ϕ, ψ that decay sufficiently rapidly, and $\langle\psi, \hat{H}\psi\rangle = \int(|\nabla\psi|^2 + V|\psi|^2)$ via integration by parts.

(a) Sketch the band diagram $\omega(k)$ that you would expect to get for this problem for eigenfunctions of the form $\psi(y)e^{ikx}$. Given an explicit formula for the analogue

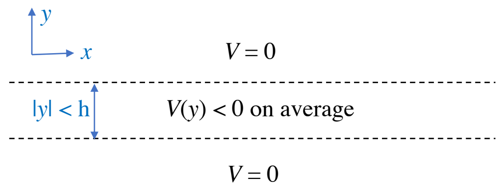

Figure 2: Schematic of an x-invariant function $V(y)$ that $= 0$ for $|y| > h$ and is otherwise "mostly" negative, i.e. $\int_{-\infty}^{\infty} V(y)dy < 0$.

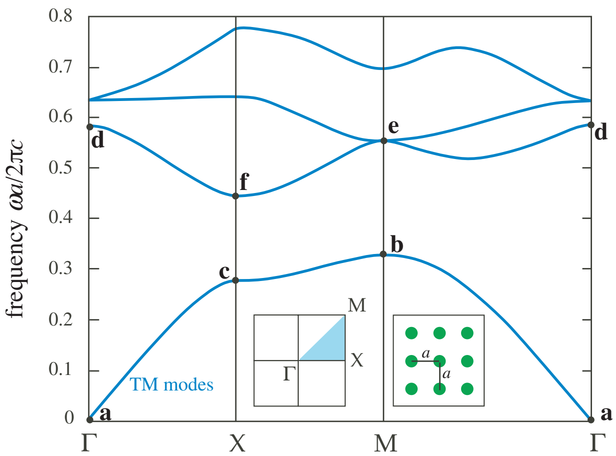

Figure 3: TM band diagram of a square lattice (lattice constant a) of circular dielectric rods (right inset) plotted around the boundary of the irreducible Brillouin zone (left inset). Various points (black dots) are labelled with letters (**a–f**) for future reference.

of the "light cone" here (the propagating solutions in the $|y| \to \infty$ regions)?

(b) The smallest ω minimizes the Rayleigh quotient $\frac{\langle \psi, \hat{H}\psi \rangle}{\langle \psi, \psi \rangle}$. Use this fact, along with a suitable trial function (similar to homework) to show that there is at least one guided mode below the light cone for every $k \neq 0$.

[Recall from homework/class Lebesgue's dominated convergence theorem: you can interchange limits and integrals for $\int f(y)$ if $|f(y)| \leq g(y)$ for some $g(y)$ with $\int g < \infty$.]

Problem 3: Projected band diagram

The TM band diagram of a square lattice (lattice constant a) of circular dielectric rods is shown in figure 3. In class, we considered linear defects along the Γ–X direction (e.g. removing a row of rods). Here, we will consider linear defects along the Γ–M (**diagonal**) direction, with period $a\sqrt{2}$ along that direction.

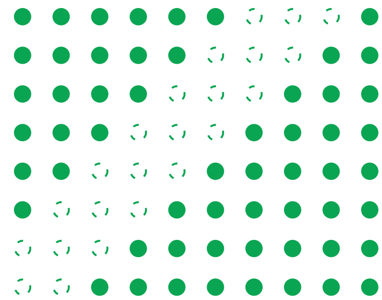

Figure 4: Linear defect in the diagonal (Γ–M) direction of a square lattice of rods formed by removing $N = 3$ adjacent diagonal rows of rods (removed rods shown as dashed outlines).

(a) **Sketch the projected band diagram** along the Γ–M direction: plot **the first two bands** of the periodic crystal as a function of the component k_d of **k** along this direction, for the irreducible Brillouin zone in k_d.

(i) Recall that the M point is $(\frac{\pi}{a}, \frac{\pi}{a})$. Given that the period is $a\sqrt{2}$, where is the edge of the Brillouin zone in k_d along the Γ–M segment?

(ii) On your plot, **label** with letters **a–f** the points corresponding to those labelled locations in figure 3.

(iii) Be sure to shade any continuous ranges of ω that occur when you project the original Brillouin zone onto your (k_d, ω) plot.

(b) **Sketch (qualitatively)** your best guess for **the projected band diagram** including the modes **of a defect** where N adjacent diagonal rows of rods are removed (e.g. as shown in figure 4 for $N = 3$). **Sketch what happens** as N increases, and in the limit as $N \to \infty$. You may assume that there are *no surface states* for this crystal termination. [Hint: it might be easier to **start with** the $N \to \infty$ limit and then sketch what happens as N decreases.]