Note on decomposing functions into partner functions

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In the representation-theory handout for 18.369, it says that any function \( \psi(\vec{x}) \) can be decomposed into a sum of partner functions of the different irreps of any symmetry group \( G \). Recall that for a coordinate transformation \( g \) (a rotation or translation), I denote the corresponding transformation of functions \( \psi \) by \( \hat{O}_g \).\(^1\) What follows is a brief proof of that.

1. Consider the set \( S = \{ \hat{O}_g \psi \text{ for all } g \in G \} \). Form a basis \( \psi_i \) of \( S \), for \( i \in \{ 1, \ldots, d \} \) where \( d \) is the dimension of the subspace spanned by \( S \) (the number of linearly independent functions in \( S \)).

2. By construction, \( \hat{O}_g \psi_j \in S \) for any \( j \in \{ 1, \ldots, d \} \), \( g \in G \). Hence \( \hat{O}_g \psi_j = \sum_{i=1}^d \psi_i D_{ij}(g) \) where \( D_{ij}(g) \) are some coefficients depending on \( i \), \( j \), and \( g \).

3. The matrices \( D(g) \) with entries \( D_{ij}(g) \) form a representation of \( G \). Proof:

\[
\hat{O}_{g_1} \hat{O}_{g_2} \psi_j = \hat{O}_{g_1g_2} \psi_j = \sum_{i=1}^d \psi_i D_{ij}(g_1g_2) \\
= \sum_{i=1}^d \psi_i \left[ \sum_{k=1}^d D_{ik}(g_1) D_{kj}(g_2) \right] \\
= \sum_{i=1}^d \sum_{k=1}^d \psi_i D_{ik}(g_1) D_{kj}(g_2).
\]

Comparing the first and last lines, which must be true for any \( i, j \), we find \( D_{ij}(g_1g_2) = \sum_{k=1}^d D_{ik}(g_1) D_{kj}(g_2) \), which is exactly the formula for a matrix multiplication, so \( D(g_1g_2) = D(g_1)D(g_2) \). Hence \( D \) is a representation.

4. \( D \) must be reducible into one or more irreps \( D^{(a)} \) of \( G \), i.e. we can perform a change of basis to \( \tilde{D} = S^{-1}DS \) that block-diagonalize \( \tilde{D} \) into irreps. Perform the same change of basis on \( \psi_i \) to obtain the corresponding basis functions \( \tilde{\psi}_j = \sum_i \psi_i S_{ij} \). By construction, the \( \tilde{\psi}_j \) are partners of \( \tilde{D} \), and hence they are partners of the irreps that \( \tilde{D} \) reduces into.

5. \( \psi \in S \) since the identity \( E \in G \), so \( \psi \) is in the span of the basis functions \( \psi_i \) and hence of \( \tilde{\psi}_i \). Hence \( \psi = \sum_i c_i \tilde{\psi}_i \) for some coefficients \( c_i \), which from above is a sum of partner functions of one or more of various irreps of \( G \). (Note it is easy to show that the partner functions of an irrep form a vector space: summing two partners of the same irrep or multiplying them by scalars \( c_i \) yields another partner function.) Q.E.D.

\(^1\)Some authors just use \( g \) interchangeably for rotations of the coordinate space or rotations of the Hilbert space, but for vector fields it is confusing if you don’t distinguish the two. In hindsight, maybe I should have used \( \hat{g} \) instead of \( \hat{O}_g \); oh well.