Note on decomposing functions into partner functions

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In the representation-theory handout for 18.369, it says that any function $\psi(\vec{x})$ can be decomposed into a sum of partner (basis) functions of the different irreps of any symmetry group G. Recall that for a coordinate transformation g (a rotation or translation), I denote the corresponding transformation of functions ψ by $\hat{g}$.[1] What follows is a brief proof of that.

1. Consider the set $S = \{\hat{g}\psi \text{ for all } g \in G\}$ and the subspace $\mathcal{S}$ spanned by S. Form a basis of $\mathcal{S}$ from d elements $\psi_i = \widehat{g_i}\psi$ of S, for $i \in \{1, \ldots, d\}$ where d is the dimension of $\mathcal{S}$ (the number of linearly independent functions in S).

2. By construction, $\hat{g}\psi_j = \hat{g}\widehat{g_i}\psi = \widehat{g'}\psi \in S, \mathcal{S}$ for any $j \in \{1, \ldots, d\}$, $g \in G$. Hence $\hat{g}\psi_j = \sum_{i=1}^{d} \psi_i D_{ij}(g)$ where $D_{ij}(g)$ are some coefficients depending on i, j, and g.

3. The matrices $D(g)$ with entries $D_{ij}(g)$ form a representation of G. Proof:

$$\begin{aligned}
\widehat{g_1}\widehat{g_2}\psi_j &= \widehat{g_1 g_2}\psi_j = \sum_{i=1}^{d} \psi_i D_{ij}(g_1 g_2) \\
&= \widehat{g_1} \sum_{k=1}^{d} \psi_k D_{kj}(g_2) = \sum_{k=1}^{d} \left[\sum_{i=1}^{d} \psi_i D_{ik}(g_1) \right] D_{kj}(g_2) \\
&= \sum_{i=1}^{d} \psi_i \left[\sum_{k=1}^{d} D_{ik}(g_1) D_{kj}(g_2) \right].
\end{aligned}$$

   Comparing the first and last lines, which must be true for any i, j, we find $D_{ij}(g_1 g_2) = \sum_{k=1}^{d} D_{ik}(g_1) D_{kj}(g_2)$, which is exactly the formula for a matrix multiplication, so $D(g_1 g_2) = D(g_1) D(g_2)$. Hence D is a representation.

4. D must be reducible into one or more irreps $D^{(\alpha)}$ of G (with dimension $d^{(\alpha)}$), i.e. we can perform a change of basis to $\tilde{D} = S^{-1} D S$ that block-diagonalize $\tilde{D}$ into irreps. Perform the same change of basis on ψ_i to obtain the corresponding basis functions $\tilde{\psi}_j^{(\alpha)}$ (where α denotes the block, i.e. the irrep, and j is the index within the block). By construction, the $\tilde{\psi}_i^{(\alpha)}$ are partners of the irrep blocks $D^{(\alpha)}$.

5. $\psi \in S$ since the identity $E \in G$, so ψ is in the span of the basis functions ψ_i and hence of $\tilde{\psi}_i^{(\alpha)}$. Hence $\psi = \sum_\alpha \sum_{i=1}^{d^{(\alpha)}} c_i^{(\alpha)} \tilde{\psi}_i^{(\alpha)}$ for some coefficients $c_i^{(\alpha)}$, which from above is a sum of partner functions of one or more of various irreps of G.[2] Q.E.D.

[1] Some authors just use g interchangeably for rotations of the coordinate space or rotations of the function space, but for vector fields it can be confusing if you don't distinguish the two. e.g. for a covariant vector field $\vec{F}(\vec{x})$, we define $\hat{g}\vec{F} = g\vec{F}(g^{-1}\vec{x})$.

[2] The same irrep may appear multiple times in the reduction of D, but in that case we can combine the respective partner functions into new partner functions (it is easy to show that summing/scaling partner functions of the same irrep yields another partner function).