The Mathematics of Lasers
from Nonlinear Eigenproblems to Linear Noise

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What is a **laser**?

- a laser is a **resonant cavity**…
- with a **gain medium**…
- pumped by external power source
  
- **population inversion** → **stimulated emission**
Resonance

an oscillating mode trapped for a long time in some volume
(of light, sound, …) lifetime $\tau \gg 2\pi/\omega_0$
frequency $\omega_0$ quality factor $Q = \omega_0\tau/2$
energy $\sim e^{-\omega_0 t/Q}$

modal volume $V$

[ Notomi et al. (2005). ]

[ Schliesser et al., PRL 97, 243905 (2006) ]

How Resonance?
need mechanism to trap light for long time

metallic cavities: good for microwave, dissipative for infrared

ring/disc/sphere resonators: a waveguide bent in circle, bending loss $\sim \exp(-\text{radius})$

photonic bandgaps (complete or partial + index-guiding)

Passive cavity (lossy)

Gain

pump = 0

Loss

Mode intensity

linear loss of passive cavity
Pump $\Rightarrow$ \textbf{Gain: nonlinear in field strength}

\begin{align*}
&\text{Gain} \\
pump &= 0.2 \\
\text{Loss} &\quad \text{Mode intensity}
\end{align*}
pump = 0.3 threshold
Loss Mode intensity

Gain

$pump = 0.4$

Loss

Mode intensity
Gain

pump = 0.5

Loss

Mode intensity
Gain

pump = 0.6

Mode intensity

Loss
The steady state

Gain

pump = 0.7

Loss

Mode intensity
goals of laser theory:
for a given laser, determine:

1) thresholds
2) field emission patterns
3) output intensity
4) frequencies

of steady-state operation

[if there is a steady state]
What’s new in SALT? Why ab initio?

Basic semiclassical theory from early 60’s and much of quantum theory

No general method for accurate solution of the equations for arbitrary resonator including non-linearity, openness, multi-mode

Direct numerical solutions in space and time impractical

SALT: direct solution for the multimode steady-state including openness, gain saturation and spatial hole-burning, arbitrary geometry

Ab Initio: Only inputs are constants describing the gain medium, quantitative agreement with brute force simulations
Motivation: Modern micro/nano lasers

Complex microcavities: micro-disks, micro-toroids, deformed disks (ARCs), PC defect mode, random...

No boundary reflection at all!

No measurable passive resonances.
Semiclassical theory

1. Maxwell’s equations (classical)

\[- \nabla \times \nabla \times (E^+) - \varepsilon_c \ddot{E}^+ = \frac{1}{\varepsilon_0} \ddot{P}^+\]

cavity dielectric

polarization of gain atoms
Semiclassical theory

1. Maxwell’s equations (classical)

\[- \nabla \times \nabla \times (\mathbf{E}^+) - \varepsilon_c \ddot{\mathbf{E}}^+ = \frac{1}{\varepsilon_0} \dot{\mathbf{P}}^+\]

- cavity dielectric
- polarization of two-level gain atoms

2. Damped oscillations of electrons in atoms (quantum)

\[\dot{\mathbf{P}}^+ = (-i\omega_a - \gamma_{\perp})\mathbf{P}^+ + \frac{1}{i\hbar} \mathbf{E}^+ D\]

- atomic frequency
- population inversion (drives oscillation)
Semiclassical theory

1. Maxwell’s equations
\[- \nabla \times \nabla \times (E^+) - \varepsilon_c E^+ = \frac{1}{\varepsilon_0} \dot{P}^+\]
cavity dielectric
polarization of two-level gain atoms

2. Damped oscillations of electrons in atoms
\[\dot{P}^+ = (-i\omega_a - \gamma_\perp)P^+ + \frac{1}{i\hbar} E^+ D\]
atomic frequency
population inversion (drives oscillation)

3. Rate equation for population inversion
\[\dot{D} = \gamma_\parallel (D_0 - D) + \frac{1}{\hbar \omega_a} \text{Re} \left[ (E^+)^* \cdot \dot{P}^+ \right]\]
\[\gamma_\perp \text{ and } \gamma_\parallel \text{ phenomenological relaxation rates (from collisions, etc)}\]
Maxwell–Bloch equations

- fully time-dependent, multiple unknown fields, nonlinear (Haken, Lamb, 1963)

\[
\begin{align*}
- \nabla \times \nabla \times (E^+) - \varepsilon_c \ddot{E}^+ &= \frac{1}{\varepsilon_0} \ddot{P}^+ \\
\dot{P}^+ &= (-i\omega_a - \gamma_\perp)P^+ + \frac{1}{i\hbar}E^+D \\
\dot{D} &= \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar}[E^+ \cdot (P^+)^* - P^+ \cdot (E^+)^*]
\end{align*}
\]

Polarization induces inversion

Inversion drives polarization
Maxwell–Bloch FDTD simulations very expensive, but doable.

Bermel et. al. (PRB 2006)
Problem: timescales!

\[ \gamma_\parallel \ll \gamma_\perp \ll \omega_a \]

FDTD takes very long time to converge to steady state

Solving Maxwell–Bloch for just one set of lasing parameters is expensive and slow, let alone design
Advantage: timescales!

\[ \frac{\gamma_{\parallel}}{\gamma_{\perp}} \ll 1, \quad \frac{\gamma_{\perp}}{\omega_a} \ll 1 \]

- hard for numerics
- good for analysis
Ansatz of steady-state modes

\[ E^+(x,t) = \sum_{\mu=1}^{M} \Psi_{\mu}(x)e^{-ik_{\mu}t}, \]

\[ P^+(x,t) = \sum_{\mu=1}^{M} p_{\mu}(x)e^{-ik_{\mu}t}, \]
Two key approximations

1. “rotating-wave approximation”
   - fast oscillation average out to zero;
   - all oscillations fast compared to inversion

\[
\begin{align*}
\dot{D} &= \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar} \left[ E^+ \cdot (P^+)^* - P^+ \cdot (E^+)^* \right]
\end{align*}
\]

… leads to…

\[
\dot{D} \approx 0
\]

2. stationary-inversion approximation
before:

\[- \nabla \times \nabla \times (E^+) - \varepsilon_c \ddot{E}^+ = \frac{1}{\varepsilon_0} \dot{P}^+\]

\[\dot{P}^+ = (-i\omega_a - \gamma_\perp)P^+ + \frac{g^2}{i\hbar}E^+ D\]

\[\dot{D} = \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar}[E^+ \cdot (P^+)^* - P^+ \cdot (E^+)^*]\]

\[\varepsilon_m(x) = \varepsilon_c(x) + \frac{\gamma_\perp}{\omega_m - \omega_a + i\gamma_\perp}\left[\frac{D_0(x)}{1 + \sum \left| \frac{\gamma_\perp}{\omega_\nu - \omega_a + i\gamma_\perp} E_\nu \right|^2}\right]\]

Still nontrivial to solve:
equation is nonlinear in both

eigenvalue $\omega_m \leftarrow$ easier

eigenvector $E_m \leftarrow$ harder

after:

Steady-State Ab-Initio Lasing Theory,
“SALT”
(Tureci, Stone, 2006)

\[\nabla \times \nabla \times E_m = \omega_m^2 \varepsilon_m E_m\]
Constant-flux “CF” basis method

Tureci, Stone, PRA 2006
(same paper that introduced SALT)

\[ E_m(x) = \sum_{n=1}^{N} c_{mn} F_n(x) \]

solutions to linear problem at threshold

\[ T(\omega_m, c_{mn})c_{mn} = 0 \]

problem still nonlinear, but very small dimensionality
Example of SALT results using CF basis method

Ge et al. (PRA 2010)
CF basis method not scalable

1. far above threshold, expansion efficiency decreases, need more basis functions
2. in most cases basis functions need to be obtained numerically
3. storage in 2d and 3d
Common pattern for theories in physics

1. purely analytic solutions (handful of cases)
2. specialized basis (problem-dependent and hard to scale to arbitrary systems)
3. generic grid/mesh, discretize

Can we solve the equations of SALT (which are nonlinear) on a grid **without an intermediate basis**?
Finite-difference discretization

\[ \nabla \times \nabla \times \mathbf{E}_m = \omega_m^2 \varepsilon_m (\omega_m, \{\mathbf{E}_\nu\}) \mathbf{E}_m \]

degrees of freedom:
\[ \mathbf{E}_m \] at every point on (Yee) grid
\[ m = 1, 2, \ldots \# \text{ modes} \]

\[ \nabla \times \nabla \times \rightarrow \text{finite differences} \]

→ “just” solve

… but is it reasonable to solve \(10^4\text{–}10^7\) coupled nonlinear equations?
Yes!

Newton: $f(v) = 0$

$v_{\text{guess}} \rightarrow v_{\text{guess}} - \left( \frac{\partial f}{\partial v} \right)^{-1} f$

$v = \begin{pmatrix} E_m \\ \omega_m \end{pmatrix}$, $f = \begin{pmatrix} \nabla \times \nabla \times + \omega_m^2 \epsilon_m \\ E_m(x_0) \end{pmatrix}$

key fact #1:
Newton’s method converges very quickly when we have a good initial guess (near the actual answer)

key fact #2:
we have a good initial guess (at threshold, the problem is linear in $E_m$, easy to solve)
The linear problem is sparse!

\[ \frac{\partial f}{\partial v} = \text{Jacobian} \quad \ldots \quad \text{sparse} \]

\[ \frac{\partial f}{\partial v} \delta v \approx -f(v_{\text{guess}}) \]

\[ Ax = b \]

A sparse = good solvers

ex: Matlab “\”, PETSc
\[ f(v) = 0 \] must have same number of equations as unknowns

1. frequency \( \omega_m \) is unknown, and

2. amplitude as separate unknown (to eliminate the trivial \( E=0 \) solution)

\[ E_m(x) = a_m E_m^{\text{normalized}}(x) \]
\[ f(v) = 0 \]

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1. frequency \( \omega_m \) is unknown, and

2. amplitude as separate unknown (to eliminate the trivial \( E=0 \) solution)

\[ E_m(x) = a_m E_m^{\text{normalized}}(x) \]

two extra unknowns, two extra equations, Jacobian matrix stays square

normalization and phase fixing (real and imaginary parts give two equations)

\[ \hat{n} \cdot E_m^{\text{normalized}}(x_0) = 1 \]
How to get initial guesses:
Increase the pump as gradually as needed

threshold: \( \text{linear in } E \)

... well known solvers

Newton converges quadratically given good initial guess
Benchmark comparison with previous 1d results

1d laser cavity

Benchmarks for ~1000 pixels
Maxwell—Bloch (FDTD)
~60 CPU hours
SALT, Direct Newton
20 CPU seconds!!!

c.f. SALT CF Basis
~5 CPU minutes
Confirmation of known 2d results

mode-switching behavior in microdisk laser (solid = Newton, dotted = basis)

field profile of mode 1
Demonstration of 3d calculation

full-vector simulation of lasing defect mode in photonic crystal slab

≈50 x 50 x 30 pixel computational cell: 10 CPU minutes on a laptop with SALT + Newton’s method
Lasers: Quick Review

laser = lossy optical resonance + nonlinear gain

threshold: increase pump until gain ≥ loss at amplitude=0

above threshold:
nonlinear gain/loss ⇒ stable amplitude

mirrors/confinement

confined mode in gain medium

“pump” energy

[ image: wikipedia ]

mode |amplitude|^2
Lasers: Quick Review

laser = lossy optical resonance + nonlinear gain

toy TCMT model of single-mode laser:

\[ \frac{da_1}{dt} = C_{11} \left( |a_1^0|^2 - |a_1|^2 \right) a_1 \Rightarrow a_1 \rightarrow a_1^0 \]

above threshold:

\[ |a_1|^2 \]

nonlinear gain/loss

\[ \Rightarrow \text{stable amplitude} \]

steady state

= zero linewidth!

(δ-function spectrum)

= toy instantaneous nonlinearity

nonlinear coefficient
Laser noise:

random (quantum/thermal) currents
“kick” the laser mode
⇒ Brownian phase drift = finite linewidth
Johnson–Nyquist Noise \[ 1926 \]
(no relation to me)

random current $I$ from thermal noise:

mean $\langle I^2 \rangle \approx 4kT \times (1/R) \times \text{(bandwidth)}$

$\longrightarrow kT$ for high $T$ (classical thermal fluctuations)
$\longrightarrow \hbar \omega/2$ for low $T$ (quantum zero-point fluctuations)

Generalization: the Fluctuation–Dissipation Theorem

$$\langle J_i(\omega, \mathbf{x}) J_j^*(\omega, \mathbf{x'}) \rangle$$

$$= \frac{1}{\pi} \delta_{ij} \delta(\mathbf{x} - \mathbf{x}')$$

$$\left[ \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2kT} \right) \right] \sigma(\omega, \mathbf{x})$$

$\Rightarrow kT$ for high $T$ (classical thermal fluctuations)
$\Rightarrow \hbar \omega/2$ for low $T$ (quantum zero-point fluctuations)

[ Callen & Welton, 1951 ]
Consequences of current fluctuations

Fluctuating currents \( \mathbf{J} \) produce fluctuating electromagnetic fields.

Fields carry:
- Momentum ⇒ Casimir forces
- Energy ⇒ thermal radiation

In a laser: \( \mathbf{J} = \) random forcing
= phase drift
= nonzero laser linewidth
Toy TCMT Laser + Noise

[ = nonlinear “van der Pol” oscillator, similar to e.g. Lax (1967) ]

lowest-order stochastic ODE:

\[
\frac{da_1}{dt} \approx C_{11} \left( |a_1^0|^2 - |a_1|^2 \right) a_1 + f_1(t)
\]

tricky part: getting \( f & C \)

linearize:

\[
a_1 = \left[ a_1^0 + \delta_1(t) \right] e^{i\phi_1(t)}
\]

\[\Rightarrow \ldots \Rightarrow \langle \phi^2 \rangle = Rt\]

Brownian (Wiener) phase

\[
\delta \text{ fluctuations } \Rightarrow \text{“thermal” background}
\]

\[
\omega \quad \omega_1
\]

Lorentzian lineshape, width \( \Delta \omega = R/2\pi \)
Laser linewidth theory: Long history

Long history of laser-linewidth theory:
• Gordon (‘55), Schawlow–Townes (‘58): linewidth ~ 1/P
• Petermann (‘79): correction for lossy cavities (complex $E$)
• Henry (‘82, ‘86): nonlinear phase/amplitude coupling enhancement
• Elsasser (‘85), Kruger (‘90): 2-mode nonlinear linewidth interactions
• Generalizations: dispersion, incomplete inversion, nonuniform…

Almost always in 1d, only considering a few corrections at a time…
(e.g. dispersion but not Henry, only homogeneous inversion…)

• Chong (2013): S-matrix combination of many previous corrections
  (but not Henry factor or inhomogeneous inversion)
  … showed that corrections are intermingled in general cavities

Can we solve the full 3d inhomogeneous problem?
Laser theory troubles:

Until recently, it’s been almost impossible to solve for laser modes > threshold in complex microcavities (not 1d-like).

difficulty: optical timescale << electron relaxation timescale

Why bother with linewidth theory if we can’t solve without noise?
SALT: *steady-state ab-initio lasing theory*

= *analytical separation of optical/electronic timescales*

[Türeci, Stone, and Collier (2006)]

SALT: “ordinary” EM eigenproblem for lasing modes

\[ \nabla \times \nabla \times \mathbf{E}_m = \omega_m^2 \varepsilon_m \mathbf{E}_m \]

with nonlinear permittivity \( \varepsilon \):

\[ \varepsilon_m = \varepsilon_c(x) + \frac{\gamma_0}{\omega_m - \omega_0 + i\gamma_0} \left[ \frac{D_0(x,d)}{1 + \sum_n a_n \frac{\gamma_0}{\omega_n - \omega_0 + i\gamma_0} \mathbf{E}_n} \right]^2 \]

(Lorentzian gain spectrum, mode amplitudes \( a_n \))

**Limitation:** until recently, SALT only solvable in 1d & simple 2d
New SALT Solvers

= accurate laser modes in new geometries


\[ \nabla \times \nabla \times E_m = \omega_m^2 \varepsilon_m E_m \]

SALT: “ordinary” EM eigenproblem

with nonlinear permittivity \( \varepsilon \)

\[ \varepsilon_m = \varepsilon_c(x) + \frac{\gamma_0}{\omega_m - \omega_0 + i\gamma_0} D_0(x,d) \sum_n \left| a_n \frac{\gamma_0}{\omega_n - \omega_0 + i\gamma_0} E_n \right|^2 \]

+ full 3d nonlinear eigensolvers & PDE solvers
Ab-initio laser-linewidth starting point

The Fluctuation–Dissipation Theorem (FDT) [Callen & Welton, 1951]

\[ \langle J_i(\omega, x)J_j^*(\omega, x') \rangle = \frac{1}{\pi} \delta_{ij} \delta(x-x') \left[ \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2kT} \right) \right] \sigma(\omega, x) \]

In a gain medium, \( \sigma < 0 \) and \( T \leq 0^- \) (\( T < 0^- = \text{incomplete inversion} \))

... these small currents randomly “kick” the SALT lasing modes \( \Rightarrow \) random (Brownian) phase drift \( \Rightarrow \) linewidth

[related starting point, albeit in greatly simplified 1d media, used by Henry (1986) ]
Real TCMT equations
[ 2-level gain medium, timescale $\gg$ optical ]

Maxwell-Bloch equations:

\[
\nabla \times \nabla \times E + \epsilon_c \ddot{E} = -\ddot{P} + F \\
\dot{P} = -i(\omega_a - i\gamma_\perp)P - i\gamma_\perp ED \\
\dot{D} = -\gamma_\parallel \left( D_p - D + \frac{i}{2}(E \cdot P^* - E^* \cdot P) \right)
\]

+ rotating-wave approx. (lasing modes dominate) … lots of algebra …

\[
\dot{a}_\mu = \sum_\nu \left[ \int dx \ c_{\mu\nu}(x) \gamma(x) \int_{-\infty}^{t} dt' e^{-\gamma(x)(t-t')} \left( a_{\nu 0}^2 - |a_\nu(t')|^2 \right) \right] a_\mu + f_\mu
\]

= non-instantaneous, multi-mode nonlinear gain
TCMT coefficients

[ 2-level gain medium, timescale >> optical ]

\[ \dot{a}_\mu = \sum_\nu \left[ \int dx c_{\mu\nu}(x) \gamma(x) \int_{-\infty}^t dt' e^{-\gamma(x)(t-t')} (a_{\nu 0}^2 - |a_\nu(t')|^2) \right] a_\mu + f_\mu \]

\[ c_{\mu\nu} = \frac{-i \omega_\mu^2 \frac{\partial \epsilon(\omega_\mu)}{\partial |a_\nu|^2} E_\mu^2}{\int dx \frac{\partial}{\partial \omega} (\omega^2 \epsilon) E_\mu^2} \]

(essentially = 1\textsuperscript{st} order perturbation theory for \( \partial \omega / \partial |a|^2 \))

Currents = forcing \( F = dJ/dt \) \( \Rightarrow \)

\[ f_\mu = \frac{i \int dx E_\mu F_\mu}{\int dx \frac{\partial}{\partial \omega} (\omega^2 \epsilon) E^2} \]
**New 1-mode Linewidth \( \Gamma \) Formula**

correct TCMT \( \Rightarrow \) plug in FDT \( \Rightarrow \) \( \langle f^2 \rangle \) \( \Rightarrow \) solve stochastic ODEs...

\[
\Gamma = \frac{\hbar \omega_0}{P} \cdot \frac{\omega_0^2 \int dx dx' \left( \varepsilon''(x)|E_0(x)|^2 \right) \left( \varepsilon''(x')|E_0(x')|^2 \right) \coth \left( \frac{\hbar \omega_0 \beta(x)}{2} \right)}{\left| \int dx E_0^2 \left( \epsilon + \frac{\omega_0}{2} \frac{\partial \epsilon}{\partial \omega_0} \right) \right|^2} \cdot (1 + \tilde{\alpha}^2)
\]

(spatially varying) incomplete inversion (\& thermal noise)

\[
\varepsilon'' = \text{Im} \varepsilon
\]

\[
\tilde{\alpha} = \frac{\text{Im} c_{11}}{\text{Re} c_{11}}
\]

\( \alpha \)-factor (spatially varying) incomplete inversion

**Schawlow–Townes inverse scaling with radiated power \( P \)**

\( \sim \) generalized Petermann factor (including “bad cavity” correction for dispersion), including incomplete inversion
New predictions:

• Fun fact: “toy” instantaneous nonlinearity gives same $\Gamma$!

• Correction from inhomogeneous incomplete inversion
  (… in general, all corrections are intermingled …)

• “Bad-cavity” (high-leakage) correction to Henry $\alpha$ factor

• Closed-form generalization to arbitrary multimode lasers
in progress...

* Validation against solution of full Maxwell–Bloch equations + thermodynamic noise (in 1d) — *A. Cerjan* (Yale)

* Design a laser (e.g. with “exceptional points”) where new *corrections* are much *larger*

* Additional corrections
  [e.g. amplified spontaneous emission (ASE) for “passive” modes just below their lasing thresholds; also “colored” noise correction for broad linewidth]

* New SALT models (e.g. semiconductor lasers…)
  ⇒ *new linewidth formulas*
Thanks!

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