18.336 Problem Set 2

Due Thursday, 9 March 2006.

Note that this problem set closely follows the notes from previous terms. (See 18.336 on OpenCourseWare for the lecture notes, which is linked to from http://math.mit.edu/~stevenj/18.336)

Problem 1: Crank-Nicolson

(a) Prove that the Crank-Nicolson scheme for $u_t = -au_x$ is unconditionally stable. The C-N scheme for the discretized solution $u(m\Delta x, n\Delta t) \approx v^n_m$ is

$$v^{n+1}_m = v^n_m - a\Delta t \left[ \frac{v^{n+1}_{m+1} - v^{n+1}_{m-1}}{2\Delta x} + (1 - \alpha) \frac{v^{n+1}_{m+1} - v^{n+1}_{m-1}}{2\Delta x} \right]$$

where $\alpha = 0.5$.

(b) For what other values of $\alpha$ is this unconditionally stable?

Problem 2: Consistency and Stability

(a) Show that the following scheme is consistent with $u_t + au_x = 0$.

$$\frac{v^{n+1}_m - v^n_m}{\Delta t} + \frac{a}{2} \left( \frac{v^{n+1}_{m+1} - v^{n+1}_{m-1}}{\Delta x} + \frac{v^n_m - v^n_{m-1}}{\Delta x} \right) = 0.$$

(b) Show that this scheme is consistent with $u_t + au_{xx} = 0$:

$$\frac{v^{n+1}_m - v^n_m}{\Delta t} + \frac{a}{2} \frac{v^{n+1}_{m+2} - 3v^n_m + 3v^n_{m-1} - v^n_{m-1}}{\Delta x^3} = 0$$

If $\nu = \Delta t/\Delta x^3$ is constant, show it is stable when $0 \leq a\nu \leq \frac{1}{4}$.

Problem 3: Instability

Use the unstable forward-time centered-space scheme (see notes, example 2, section 1.3) to solve $u_t = -au_x$ on the interval $[-1, 3]$, with periodic boundary conditions $u(x, t) = u(x + 4, t)$, for the following three sets of initial data:

- $u_1(x, 0) = \sin x$
- $u_2(x, 0) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$
- $u_3(x, 0) = \sin(\pi x)$

Use $a = 1$, $\Delta x = 0.1$, and $\Delta t/\Delta x = \lambda = 0.8$.

(a) For each initial condition, plot $u(x, t)$ versus $x$ for $0 \leq t \leq 1$ (one plot per initial condition, with one curve per time step).

(b) Compute & plot the $L_2$ norm for $0 \leq t \leq 10$ on a semilog scale, show that each one diverges as $g^n$ for some constant $g$ ($n = \text{time step}$).

(c) Predict $g$ from Von-Neumann analysis as the class/notes, and compare.
(d) Which initial condition diverges most quickly, and why?

**Hint:** Given a vector $u$ in Matlab that stores $u(x, t)$ for some $t$ and $x = -1, -1 + \Delta x, \ldots, 3 - \Delta x$, you should be able to implement the time-step $u(x, t) \rightarrow u(x, t + \Delta t)$ via the Matlab command:

$$u = u + C \times ([u(2:end),u(1)] - [u(end),u(1:end-1)]);$$

for some constant $C$. 