18.335 Midterm, Spring 2015

Problem 1: (10+(10+10) points)

(a) Suppose you have a forwards-stable algorithm \( f \) to compute \( f(x) \in \mathbb{R} \) for \( x \in \mathbb{R} \), i.e. \( \| f(x) - f(x) \| = \| f \| O(\varepsilon_{\text{mach}}) \). Suppose \( f \) is bounded below and analytic (has a convergent Taylor series) everywhere; suppose it has some global minimum \( f_{\min} > 0 \) at \( x_{\min} \). Suppose that when computed in the obvious loop \( \tilde{g} \), and re-use them as needed to multiply by \( Q \) or \( Q^* \).

That is, each of the above three algorithms computes the QR factorization of \( A \)—for each of the three algorithms it is an improvement to compute \( \tilde{Q}^*b \) via that algorithm on \( \tilde{A} \) compared with computing \( \tilde{Q} \) (or its equivalent) by that algorithm and then performing the \( \tilde{Q}^*b \) multiplication?

Problem 3: (10+20+10 points)

Suppose \( A \) and \( B \) are \( m \times m \) matrices, \( A = A^* \), \( B = B^* \), and \( B \) is positive-definite. Consider the “generalized” eigenproblem of finding solutions \( x \neq 0 \) and \( \lambda \) to \( Ax = \lambda Bx \), or equivalently solve the ordinary eigenproblem \( B^{-1}Ax = \lambda x \). (In general, \( B^{-1}A \) is not
Hermitian.) Suppose that there are \( m \) distinct eigenvalues \( |\lambda_1| > |\lambda_2| > \cdots > |\lambda_m| \) and corresponding eigenvectors \( x_1, \ldots, x_m \).

(a) Show that the \( \lambda_k \) are real and that \( x_i^* B x_j = 0 \) for \( i \neq j \). (Hint: multiply both sides of \( Ax = \lambda B x \) by \( x^* \), similar to the derivation for Hermitian problems in class.)

(b) Explain how to generalize the modified Gram–Schmidt algorithm (figure 1) to compute an “SR” factorization \( B^{-1} A = S R \) where \( S^* BS = I \). (That is, the columns \( s_k \) of \( S \) form a basis for the columns of \( B^{-1} A \) as in QR, but orthogonalized so that \( s_i^* B s_j = 0 \) for \( i \neq j \) and \( = 1 \) for \( i = j \).) Make sure your algorithm still requires \( \Theta(m^3) \) operations!

(c) In exact arithmetic, what would \( S \) in the SR factorization of \( (B^{-1} A)^k \) converge to as \( k \to \infty \), and why? (Assume the “generic” case where none of the eigenvectors happen to be orthogonal to the columns of \( B \).)