18.335 Problem Set 3

Due Friday, 7 October 2011.

Problem 1: QR and orthogonal bases

(a) Prove that $A = QR$ and $B = RQ$ have the same eigenvalues, assuming A is a square matrix. Then do a little experiment: Construct a random 10×10 real-symmetric matrix in Matlab via `X=rand(10,10);` `A = X' * X`. Use `[Q,R] = qr(A)` to compute the QR factorization of A, and then compute $B = RQ$. Then find the QR factorization $B = Q'R'$, and compute $R'Q'$...repeat this process until the matrix converges. From what it converges to, suggest a procedure to compute the eigenvalues and eigenvectors of a matrix (no need to prove that it converges in general).

(b) The book suggests this procedure to solve least-square problems $A^*Ax = A^*b$: instead of computing $A = \hat{Q}\hat{R}$ and then solving $\hat{R}x = \hat{Q}^*b$, you should instead form the matrix $B = (A, b)$ with $n+1$ columns, do QR factorization to get $B = \hat{Q}'\hat{R}'$ and then solve $\hat{R}'_{1:n,1:n}x = \hat{R}'_{1:n,n+1}$. That is, it claims that the first n columns of $\hat{R}'$ give $\hat{R}$, and that the last column of $\hat{R}'$ gives $\hat{Q}^*b$. Explain why this must be true, regardless of what QR algorithm is used (don't use the specific structure of the Householder algorithm as the book does, use the definition of the QR factorization).

(c) Trefethen, problem 7.2.

(d) Trefethen, problem 10.4.

Problem 2: Schur fine

In class, we showed that any square $m \times m$ matrix A can be factorized as $A = QTQ^*$, where Q is unitary and T is an upper-triangular matrix (with the same eigenvalues as T).

(a) A is called "normal" if $AA^* = A^*A$. Show that this implies $TT^* = T^*T$. From this, show that T must be diagonal. Hence, any normal matrix (e.g. unitary or Hermitian matrices) must be unitarily diagonalizable. Hint: consider the diagonal entries of TT^* and T^*T, starting from the (1,1) entries and proceeding diagonally downwards by induction.

(b) Given the Schur factorization of an arbitary A (not necessarily normal), describe an algorithm to find the eigenvalues and eigenvectors of A, assuming for simplicity that all the eigenvalues are distinct. The flop count should be asymptotically $Km^3 + O(m^2)$; give the constant K.