18.335 Problem Set 3

Due Friday, 10 October 2008.

Problem 1: Least-squares

In class, I claimed that the solution(s) x to the "normal" equations

$$A^*Ax = A^*b$$

minimizes

$$f(x) = \|Ax-b\|_2 = x^*A^*Ax - b^*Ax - x^*A^*b + b^*b$$

over all complex vectors $x \in \mathbb{C}^n$.

(a) Prove that the solution to the normal equations is indeed an extremum of $f(x)$, by looking at the partial derivatives of $f(x)$ with respect to the real and imaginary parts of each component of x.

(b) Prove that this solution achieves minimum value of $f(x)$.

Problem 2: SVD and low-rank approximations

(a) Trefethen, probem 4.5.

(b) Trefethen, problem 5.2.

(c) Take any grayscale photograph (either one of your own, or off the web). Scale it down to be no more than 1500×1500 (but not necessarily square), and read it into Matlab as a matrix A with the `imread` command (type "`doc imread`" for instructions).

(i) Compute the SVD of A (Matlab's svd command) and plot the singular values (e.g. as a histogram, possibly on a log scale) to show the distribution.

(ii) Compute a lower-rank approximation of A by taking only the largest ν singular values for some ν (as in theorem 5.8). You can save this approximation as an image using `imwrite`, or you can plot it directly using the `pcolor` command [set `colormap('gray')` to get a grayscale image]. How big does ν have to be to get a reasonably recognizable image?

Problem 3: QR and orthogonal bases

(a) Prove that $A = QR$ and $B = RQ$ have the same eigenvalues, assuming A is a square matrix. Construct a random 10×10 real-symmetric matrix in Matlab via `X=rand(10,10); A = X' * X`. Use `[Q,R] = qr(A)` to compute the QR factorization of A, and then compute $B = RQ$. Then find the QR factorization $B = Q'R'$, and compute $R'Q'$...repeat this process until the matrix converges. From what it converges to, suggest a procedure to compute the eigenvalues and eigenvectors of a matrix (no need to prove that it converges in general).

(b) Trefethen, problem 7.2.

(c) Trefethen, problem 10.4.