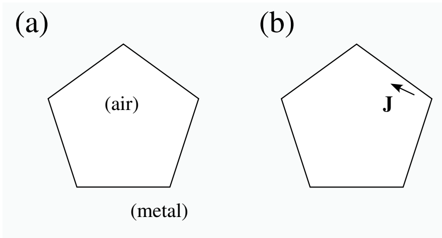

Figure 1: **(a)** A pentagonal air cavity surrounded by metal (2d, xy plane). **(b)** The same, with a point current $\mathbf{J}$ at the indicated location (with $\mathbf{J}$ pointing in the indicated direction).

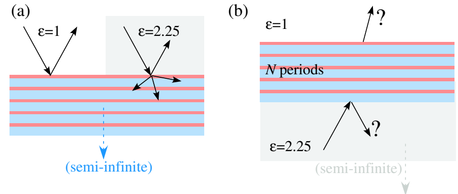

Figure 2: **(a)** Schematic of a multilayer mirror—we suppose that our structure is an omnidirectional reflector with respect to incident light from air ($\varepsilon = 1$) , but not for $\varepsilon = 2.25$ (for which there is some reflection and some transmitted beam(s)). **(b)** We have $\varepsilon = 2.25$, then N periods of the mirror, then $\varepsilon = 1$, and want to know whether (and, qualitatively, how much) light is transmitted through to the $\varepsilon = 1$.

18.325 Problem Set 5

Due Tuesday, 8 November.

Problem 1

Suppose that we have a 2d metallic electromagnetic cavity in the shape of a regular pentagon (five equal sides and angles) as shown in Fig. 1(a).

(a) Give the space group, conjugacy classes, and character table for this system.

Hint: one irreducible representation for this group ends up being the simple 2×2 coordinate rotation/reflection matrices; e.g. a rotation by θ corresponds to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Also note that $\cos(\frac{2\pi}{5}) = \frac{\sqrt{5}-1}{4} \approx 0.30902$ and $\cos(\frac{4\pi}{5}) = -\frac{1+\sqrt{5}}{4} \approx -0.80902$.

(b) Suppose that we have a point current $\mathbf{J}$ as shown in Fig. 1(b). (That is, $\mathbf{J}(x, y)$ is given by a Dirac delta function with the vector $\mathbf{J}$ lying in the xy plane.) Decompose this into partner functions of your irreducible representations from above, using the projection operators:

$$\hat{P}^{(\alpha)} = \frac{d_\alpha}{|G|} \sum_g \chi^{(\alpha)}(g)^* \hat{O}_g$$

(The result should be a sketch, expressing the original sketch as a sum of several pentagon sketches with arrows at appropriate places in the pentagons. Don't try too hard to get the *length* of the arrows to scale; just label them with "1/2", "2", etcetera according to whether their length is half, twice, etcetera of the original current amplitude.)

(c) Show how you can place a *single* point current $\mathbf{J}$ to excite fields transforming as a *single* one-dimensional irreducible representation. Show how you can place a *different* single point current $\mathbf{J}$ to excite fields with a single *two*-dimensional irreducible representation.

Problem 2

(a) We have a certain multilayer film consisting of two repeating layers of ε_1 and ε_2. Suppose that this structure, repeated semi-infinitely, *does* form an omnidirectional reflector (all polarizations and angles) at a frequency ω from an ambient medium $\varepsilon = 1$, but *does not* form an omnidirectional reflector at ω from an ambient medium $\varepsilon = 2.25$, as indicated schematically in Fig. 2(a). Now, suppose that we have semi-infinite $\varepsilon = 2.25$ followed by N bilayers of our periodic structure followed by semi-infinite $\varepsilon = 1$, as depicted in Fig. 2(b). If a frequency-ω planewave is incident on the layers from the $\varepsilon = 2.25$ medium, how does the reflected light and transmitted (into the $\varepsilon = 1$) light depend on N, angle, and polarization? (A quali-

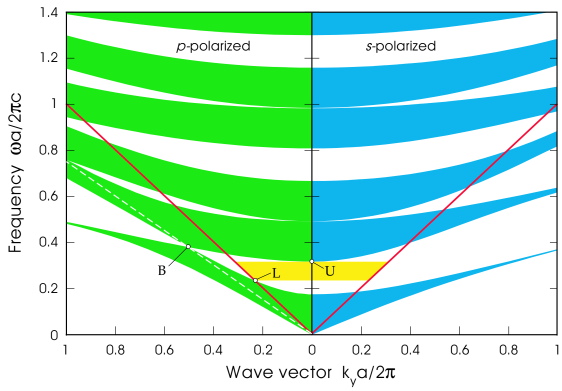

Figure 3: Projected band diagram for a multilayer film vs. surface-parallel k_y. TE ("p") polarization on the left, and TM ("s") polarization on the right.This structure is an omnidirectional reflector from the frequencies L to U with respect to air ($\varepsilon = 1$). Light line of air is shown in red.

tative description is sufficient, but must be clearly justified.)

(b) More specifically, suppose that the periodic structure has the projected band diagram shown in Fig. 3. Now, suppose we periodically *corrugate* the surface layer as shown in Fig. 4, parallel to the interface, with a period Λ (uniform in the z direction, out of plane).

(i) If we consider only planewaves incident (from air) in the xy plane as depicted in Fig. 4, for what Λ or Λ's is the omni-directional reflection property maintained? (If you need to refer to a particular numerical value ω or k_y in Fig. 3, you can simply label the corresponding point on Fig. 3 rather than trying to work out what the number is.)

(ii) What about if we consider planewaves that are not in the xy plane (i.e. they have some out-of-plane $\mathbf{k}$ component)? Is there any (finite, nonzero) Λ for which omnidirectional reflection is preserved for *all* three-dimensional angles and polarizations? Why or why not?

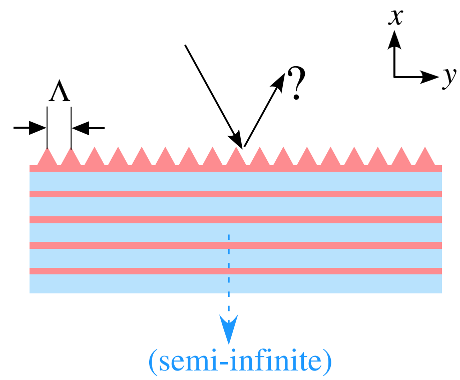

Figure 4: Reflection(?) from a multilayer film whose surface is periodically corrugated with period Λ along the y direction. Light is incident from above (air) with some angle and polarization.

Problem 3

Suppose that we have a 4×4 Hermitian matrix O (that is, Hermitian under the usual inner product $\langle x|y\rangle = \sum_n x_n^* y_n$ for 4×1 vectors $|x\rangle$ and $|y\rangle$), with eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

and corresponding eigenvalues: 1, 2, -1, -1.

(a) Now, suppose that we change O by the small ΔO:

$$\Delta O = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\delta \ll 1$. Give the approximate change in the eigenvalues (to first order in δ), and the approximate new eigenvectors (to zero-th order in δ).

(b) Are your approximate new *largest* and *smallest* eigenvalues *larger* or *smaller* than the corresponding *exact* eigenvalues? Why? (*Don't* try to compute the exact eigenvalues, which would require you to find the roots of a quartic equation.)

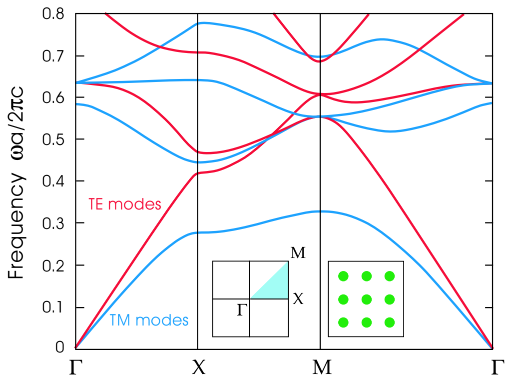

Figure 5: TE (red) and TM (blue) band diagram of a square lattice of dielectric cylinders (inset) in air, around the edges of the irreducible Brillouin zone (inset).

Problem 4

Consider the square lattice of dielectric rods, whose band diagram and irreducible Brillouin zone are shown in Fig. 5. Now, suppose that we change the shape of the rods from cylindrical to *pentagonal* (with the pentagons oriented "point-upward" as in Fig. 1, keeping the area of the rods the same (you can assume that this is therefore a small perturbation).

(a) What is the new irreducible Brillouin zone?

(b) Will any degeneracies remain in the band diagram? If so, which?

(c) Consider a solution in the perturbed (pentagonal) structure with $\mathbf{k}$ on the line segment between Γ and X (exclusive, i.e. $\mathbf{k} \neq \Gamma, X$). In what direction is the solution's electromagnetic energy propagating (or directions, depending on the band)? What about for $\mathbf{k}$ between Γ and M?

Problem 5

(a) Suppose we have a semi-infinite 1d periodic structure of ε_1 and ε_2 bounded by $\varepsilon = 1$ on one side, we look at its TM modes only, and we terminate it (with half of the high-index layer) so that it has a single surface state in the first gap. Sketch the TM projected band diagram as a function of the surface-parallel k_y (up to the first gap, *i.e.* the first two continuum regions, is enough). (That is, the structure looks like the top half of Fig. .)

(b) Now, suppose that we have *two* such semi-infinite structures as shown in Fig. 1, *both* terminated with half a high-index layer so that they would, by themselves, have surface states. In between them is air (separated by $\varepsilon = 1 < \varepsilon_1, \varepsilon_2$). Sketch what the TM projected band diagram looks like when they are some large but finite distance from one another, and sketch how this changes as they get closer to one another. Clearly indicate what happens to the surface states.

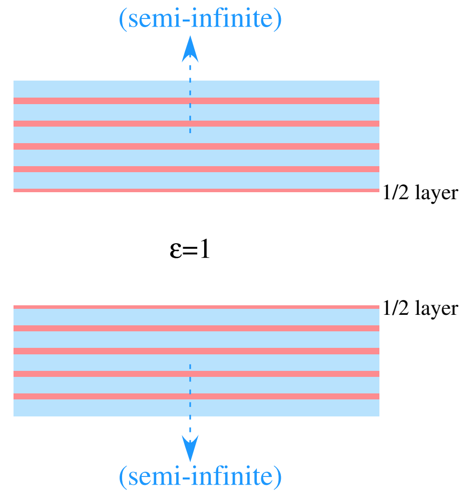

Figure 6: Two semi-infinite multilayer mirror separated by some distance of air ($\varepsilon = 1$, less than both of the mirror materials). Each film is terminated with half a high-index layer, which for an isolated semi-infinite structure would support a single TM surface state in the first gap.