0 Review

Suppose we have some vector space $V$ of functions $u(x)$ on a domain $\Omega$, an inner product $\langle u, v \rangle$, and a linear operator $\hat{A}$. [More specifically, $V$ forms a Sobolev space, in that we require $\langle u, Au \rangle$ to be finite.] $\hat{A}$ is self-adjoint if $\langle u, \hat{A}v \rangle = \langle \hat{A}u, v \rangle$ for all $u, v \in V$, in which case its eigenvalues $\lambda_n$ are real and its eigenfunctions $u_n(x)$ can be chosen orthonormal. $\hat{A}$ is positive definite (or semidefinite) if $\langle u, \hat{A}u \rangle > 0$ (or $\geq 0$) for all $u \neq 0$, in which case its eigenvalues are $> 0$ (or $\geq 0$); suppose that we order them as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Suppose that $\hat{A}$ is positive definite, so that $N(\hat{A}) = \{0\}$ and $\hat{A}u = f$ has a unique solution for all $f$ in some suitable space of functions $C(\hat{A})$. Then, for scalar-valued functions $u$ and $f$, we can typically write

$$u(x) = \hat{A}^{-1}f = \int_{x' \in \Omega} G(x, x') f(x') \, \text{d}x'$$

in terms of a Green’s function $G(x, x')$, where $\int_{x' \in \Omega}$ denotes integration over $x'$. In this note, we don’t address how to find $G$, but instead ask what properties it must have from the self-adjointness and definiteness of $\hat{A}$. [This generalizes in a straightforward way to vector-valued $u(x)$ and $f(x)$, in which case $G(x, x')$ is matrix-valued.]

1 Self-adjointness of $\hat{A}^{-1}$ and reciprocity of $G$

We can show that $(\hat{A}^{-1})^* = (\hat{A}^*)^{-1}$, from which it follows that if $\hat{A} = \hat{A}^*$ ($\hat{A}$ is self-adjoint) then $\hat{A}^{-1}$ is also self-adjoint. In particular, consider $\hat{A}^{-1}\hat{A} = 1$: $\langle u, v \rangle = \langle u, \hat{A}^{-1}\hat{A}v \rangle = \langle \hat{A}^{-1}u, \hat{A}v \rangle = \langle \hat{A}^*(\hat{A}^{-1})^*u, v \rangle$, hence $\hat{A}^*(\hat{A}^{-1})^* = 1$ and $(\hat{A}^{-1})^* = (\hat{A}^*)^{-1}$. And of course, we already knew that the eigenvalues of $\hat{A}^{-1}$ are $\lambda_n^{-1}$ and the eigenfunctions are $u_n(x)$.

What are the consequences of self-adjointness for $G$? Suppose the $u$ are scalar functions, and that the inner product is of the form $\langle u, v \rangle = \int_{\Omega} w \bar{u}v$ for some weight $w(x) > 0$. From the fact that $\langle u, \hat{A}^{-1}v \rangle = \langle \hat{A}^{-1}u, v \rangle$, substituting equation (1), we must therefore have:

$$\langle u, \hat{A}^{-1}v \rangle = \int_{x, x' \in \Omega} w(x) \bar{u(x)} G(x, x') v(x')$$

$$= \langle \hat{A}^{-1}u, v \rangle$$

$$= \int_{x, x' \in \Omega} w(x) \bar{G(x, x')} u(x') v(x') = \int_{x, x' \in \Omega} w(x') \bar{u(x)} G(x', x) v(x'),$$

where in the last step we have interchanged/relabeled $x \leftrightarrow x'$. Since this must be true for all $u$ and $v$, it follows that

$$w(x) G(x, x') = w(x') G(x', x)$$

for all $x, x'$. This property of $G$ (or its analogues in other systems) is sometimes called reciprocity. In the common case where $w = 1$ and $\hat{A}$ and $G$ are real (so that the complex conjugation can be omitted), it says that the effect at $x$ from a source at $x'$ is the same as the effect at $x'$ from a source at $x$. 

18.303: Self-adjointness (reciprocity) and definiteness (positivity) in Green’s functions

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There are many interesting consequences of reciprocity. For example, its analogue in linear electrical circuits says that the current at one place created by a voltage at another is the same as if the locations of the current and voltage are swapped. Or, for antennas, the analogous theorem says that a given antenna works equally well as a transmitter or a receiver.

1.1 Example: $\hat{A} = -\frac{d^2}{dx^2}$ on $\Omega = [0, L]$

For this simple example (where $\hat{A}$ is self-adjoint under $\langle u, v \rangle = \int \bar{u}v$), with Dirichlet boundaries, we previously obtained a Green’s function,

$$G(x, x') = \begin{cases} 
(1 - \frac{x}{L}) x < x' \\
(1 - \frac{x'}{L}) x \geq x' \end{cases}$$

which obviously obeys the $G(x, x') = G(x', x)$ reciprocity relation.

2 Positive-definiteness of $\hat{A}^{-1}$ and positivity of $G$

Not only is $\hat{A}^{-1}$ self-adjoint, but since its eigenvalues are the inverses $\lambda_n^{-1}$ of the eigenvalues of $\hat{A}$, then if $\hat{A}$ is positive-definite ($\lambda_n > 0$) then $\hat{A}^{-1}$ is also positive-definite ($\lambda_n^{-1} > 0$). From another perspective, if $\hat{A}u = f$, then positive-definiteness of $\hat{A}$ means that $0 < \langle u, \hat{A}u \rangle = \langle u, f \rangle = (\hat{A}^{-1}f, f) = (f, \hat{A}^{-1}f)$ for $u \neq 0 \iff f \neq 0$, hence $\hat{A}^{-1}$ is positive-definite. (And if $\hat{A}$ is a PDE operator with an ascending sequence of unbounded eigenvalues, then the eigenvalues of $\hat{A}^{-1}$ are a descending sequence $\lambda_1^{-1} > \lambda_2^{-1} > \cdots > 0$ that approaches 0 asymptotically from above.)

If $\hat{A}$ is a real operator (real $u$ give real $\hat{A}u$), then $\hat{A}^{-1}$ should also be a real operator (real $f$ give real $u = \hat{A}^{-1}f$). Furthermore, under fairly general conditions for real positive-definite (elliptic) PDE operators $\hat{A}$, especially for second-derivative (“order 2”) operators, then one can often show $G(x, x') > 0$ (except of course for $x$ or $x'$ at the boundaries, where $G$ vanishes for Dirichlet conditions). The analogous fact for matrices $\hat{A}$ is that if $\hat{A}$ is real-symmetric positive-definite and it has off-diagonal entries $\leq 0$ — like our $-\nabla^2$ second-derivative matrices (recall the $-1, 2, -1$ sequences in the rows) and related finite-difference matrices — it is called a Stieltjes matrix, and such matrices can be shown to have inverses with nonnegative entries.

2.1 Example: $\hat{A} = -\nabla^2$ with $u|_{\partial\Omega} = 0$

Physically, the positive-definite problem $-\nabla^2 u = f$ can be thought of as the displacement $u$ in response to an applied pressure $f$, where the Dirichlet boundary conditions correspond to a material pinned at the edges. The Green’s function $G(x, x')$ is the limit of the displacement $u$ in response to a force concentrated at a single point $x'$. The Green’s function $G(x, x')$ for some example points $x'$ is shown for a 1d domain $\Omega = [0, 1]$ in figure 1(left) (a “stretched string”), and for a 2d domain $\Omega = [-1, 1] \times [-1, 1]$ in figure 1(right) (a “square drum”). As expected, $G > 0$ everywhere except at the edges where it is zero: the whole string/membrane moves in the positive/upwards direction in response to a positive/upwards force.

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1 Such $\hat{A}^{-1}$ integral operators are typically what are called “compact” operators. Functional analysis books often prove diagonalizability (a “spectral theorem”) for compact operators first and only later consider diagonalizability of PDE-like operators by viewing them as the inverses of compact operators.


3 There are many books with “nonnegative matrices” in their titles that cover this fact, usually as a special case of a more general class of something called “M matrices,” but I haven’t yet found an elementary presentation at an 18.06 level. Note that the diagonal entries of a positive-definite matrix $P$ are always positive, thanks to the fact that $P_{ii} = e_i^T Pe_i > 0$ where $e_i$ is the unit vector in the $i$-th coordinate.
Figure 1: Examples illustrating the positivity of the Green’s function $G(x, x')$ for a positive-definite operator $(-\nabla^2$ with Dirichlet boundaries). Left: a “stretched string” 1d domain $[0, 1]$. Right: a “stretched square drum” 2d domain $[-1, 1] \times [-1, 1]$. 