18.303 Problem Set 5 Solutions

Problem 1: \((10 + 10 + 10 + 10)\)

See also the IJulia notebook posted with the solutions.

(a) Setting the slopes to be zero at \(R_1\) and \(R_2\) simply gives

\[
\alpha J'_m(kr) + \beta Y'_m(kr) = 0
\]

at the two radii, or \(E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0\) where

\[
E = \begin{pmatrix}
J'_m(kR_1) & Y'_m(kR_1) \\
J'_m(kR_2) & Y'_m(kR_2)
\end{pmatrix}
\]

Hence, writing \(f_m(k) = \det E\), we get

\[
f_m(k) = J'_m(kR_1)Y'_m(kR_2) - J'_m(kR_2)Y'_m(kR_1)
\]

Given a \(k\) for which \(f_m(k) = 0\), then we can solve for the nullspace of \(E\) by arbitrarily choosing a scaling such that \(\alpha = 1\) and solving for \(\beta\) from the first or second rows of \(E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0\):

\[
\beta = \frac{J'_m(kR_1)}{Y'_m(kR_1)} = -\frac{J'_m(kR_2)}{Y'_m(kR_2)}.
\]

(b) Note that \(f_m(k)\) for \(m > 0\) has a divergence as \(k \to 0\), so we used the ylim command to rescale the vertical axis (otherwise it would be hard to read the plot!); see the solution IJulia notebook.

(c) We’ll use the Scilab newton function, similar to class, to find the roots, with initial guesses provided by our plot in the notebook. We find \(k_1 \approx 3.196578\), \(k_2 \approx 6.31234951\), and \(k_3 \approx 9.444449\). See the solutions notebook.

(d) See the IJulia notebook. Using our \(k_1\) and \(k_2\) from part (c) and our \(\alpha\) and \(\beta\) from part (a), we find that

\[
\int_{R_1}^{R_2} ru_{0,1}(r)u_{0,2}(r)dr \approx 10^{-15},
\]

which is zero up to roundoff errors.