18.303 Problem Set 4 Solutions

Problem 1: (5+10+10 points)
In class, we defined the Kronecker product $A \otimes B$ of two matrices as the matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_{ij}$ is the $(i, j)$ entry of $A$. Derive the following properties of Kronecker products from this definition:

(a) We have

$$(A \otimes B)^* = \begin{pmatrix} \overline{a_{11}}B^* & \overline{a_{12}}B^* & \cdots \\ \overline{a_{12}}B^* & \overline{a_{22}}B^* & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

by swapping rows and columns of $A \otimes B$ and conjugating. By inspection, this is the same as $A^* \otimes B^*$, since the entries of $A^*$ are

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \cdots \\ \overline{a_{12}} & \overline{a_{22}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

(b) We have

$$(A \otimes B)(C \otimes D) = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_{11}D & c_{12}D & \cdots \\ c_{21}D & c_{22}D & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

As shown in class, when we multiply the two “block” matrices like this, we can use the ordinary “row times column” matrix-multiplication formula where we multiply blocks and add them up, i.e. the product is

$$\begin{pmatrix} \sum_{k=1}^{n} a_{1k}Bc_{k1}D & \sum_{k=1}^{n} a_{1k}Bc_{k2}D & \cdots \\ \sum_{k=1}^{n} a_{2k}Bc_{k1}D & \sum_{k=1}^{n} a_{2k}Bc_{k2}D & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where $n$ is the number of columns of $A$ (and rows of $C$). That is, the $(i, j)$-th block is

$$\sum_{k=1}^{n} a_{ik}Bc_{kj}D = \left(\sum_{k=1}^{n} a_{ik}c_{kj}\right)BD = (AC)_{ij}BD$$

where we have noticed that $\sum_{k=1}^{n} a_{ik}c_{kj}$ is simply the formula for the $i, j$ element of $AC$. But this means

$$(A \otimes B)(C \otimes D) = \begin{pmatrix} (AC)_{11}BD & (AC)_{12}BD & \cdots \\ (AC)_{21}BD & (AC)_{22}BD & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = (AC) \otimes (BD).$$

Q.E.D.
(c) Consider the vector \( y_n \otimes x_m \). Applying linearity and the mixed-product formula from the previous part, we have
\[
(I_N \otimes A + B \otimes I_M)(y_n \otimes x_m) = (I_Ny_n) \otimes (Ax_m) + (By_n) \otimes (I_Mx_m) \\
= y_n \otimes (\lambda_m x_m) + (\mu_n y_n) \otimes x_m \\
= (\lambda_m + \mu_n)y_n \otimes x_m,
\]

hence this is a “separable” eigenvector of \( I_N \otimes A + B \otimes I_M \) with eigenvalue \( \lambda_m + \mu_n \). There are \( MN \) of these \( y_n \otimes x_m \) eigenvectors, and \( I_N \otimes A + B \otimes I_M \) is \( MN \times MN \), so that is all of the eigenvectors and eigenvalues.

As discussed in class, an \( MN \)-row column vector \( y_n \otimes x_m \) can be thought of as a “two-dimensional \( M \times N \) array” that has been written in column-major order, and the matrix \( I_N \otimes A + B \otimes I_M \) can be thought of as a “two-dimensional” operator that acts with \( A \) in the \( M \) direction and \( B \) in the \( N \) direction. If we reverse this “one-dimensionalization” process, \( y_n \otimes x_m \) corresponds to the “two-dimensional array” \( x_m y_n^T \), which varies like \( x_m \) in the \( M \) direction and like \( y_n \) in the \( N \) direction. This is exactly the analogue of a 2d separable PDE solution \( X(x)Y(y) \) that is a product of one-dimensional functions \( X(x) \) and \( Y(y) \) along each direction.

**Problem 2: (5+10+5+5+(5+5+5)+5 points)**

Often, separability of the solutions is a consequence of symmetry. In this problem, you will show a related property for the case of discrete translational symmetry: a PDE that is invariant under rotation by \( 2\pi/N \). In particular, suppose that we have the circular system of \( N \) springs and masses, with identical spring constants \( k \), depicted in Figure 1. Suppose that the equation of motion of the \( n \)-th mass is
\[
m\ddot{\phi}_n = \kappa(\phi_{n+1} - \phi_n) - \kappa(\phi_n - \phi_{n-1}).
\]

(a) Since \( \ddot{\phi}_n = \frac{\kappa}{m}(\phi_{n+1} - 2\phi_n + \phi_{n-1}) \), we can write
\[
A = \frac{\kappa}{m} \begin{pmatrix}
-2 & 1 & & & 1 \\
1 & -2 & 1 & & \\
& 1 & -2 & 1 & \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -2 & 1 \\
& & & & 1 & -2
\end{pmatrix}.
\]

Note the first and last rows! This is a consequence of the periodicity of the system, since we can identify \( \phi_0 = \phi_N \) and \( \phi_{N+1} = \phi_1 \).

(b) To check definiteness, the easiest way is to factorize \( A \). Similar to class, we write \( \ddot{\phi}_n \) in two steps: first we compute \( \psi_{n+0.5} = \phi_{n+1} - \phi_n \), then we compute \( \dot{\phi}_n = \frac{\kappa}{m}(\psi_{n+0.5} - \psi_{n-0.5}) \). Unlike the 1d case in class, however, there are only \( N \) values \( \psi_{n+0.5} \), equal to the number of springs! Hence, we obtain an \( N \times N \) matrix \( D \) given by:
\[
\begin{pmatrix}
\psi_{1.5} \\
\psi_{2.5} \\
\vdots \\
\psi_{N-0.5} \\
\psi_{N+0.5}
\end{pmatrix} = Dx =
\begin{pmatrix}
-1 & 1 & & & 1 \\
1 & -1 & 1 & & \\
& 1 & -1 & 1 & \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -1 & 1 \\
& & & & 1 & -1
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_{N-1} \\
\phi_N
\end{pmatrix},
\]
where we must be careful to get the periodicity right for the last row \( \psi_{N+0.5} = \phi_1 - \phi_N \).

Similarly, noting that \( \dot{\phi}_1 = \frac{\kappa}{m} (\psi_{1.5} - \psi_{N+0.5}) \), we have:

\[
\ddot{x} = \frac{\kappa}{m} \begin{pmatrix}
1 & 0 & \cdots & 0 & -1 \\
-1 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & -1 & 0 \\
0 & \cdots & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
\psi_{1.5} \\
\psi_{2.5} \\
\vdots \\
\psi_{N-0.5} \\
\psi_{N+0.5}
\end{pmatrix} = -\frac{\kappa}{m} D^T D x,
\]

where we have identified that the matrix to take the differences of the \( \psi_{n+0.5} \) is precisely \(-D^T\). Hence, \( A = -\frac{\kappa}{m} D^T D \), which by inspection is at least \textbf{negative semidefinite} (from class).

It is not negative-definite, however. This can be checked in a variety of ways, most easily by noticing that

\[
D \begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
1
\end{pmatrix} = 0,
\]

and hence \( D \) is not full-rank (and similarly for \( A \)).

(c) Multiplying \( RA \) acts \( R \) on each of the columns of \( A \), i.e. it permutes each column, giving:

\[
RA = \frac{\kappa}{m} \begin{pmatrix}
1 & -2 & 1 \\
1 & -2 & 1 \\
\vdots & \ddots & \ddots \\
1 & -2 & 1 \\
1 & -2 & 1
\end{pmatrix}.
\]

Multiplying \( AR = (R^T A^T)^T = (R^T A)^T \) is equivalent to permuting each row of \( A \) by \( R^T \) (i.e. in the opposite direction), hence

\[
R^T A = \frac{\kappa}{m} \begin{pmatrix}
1 & -2 & 1 \\
1 & -2 & 1 \\
\vdots & \ddots & \ddots \\
1 & -2 & 1 \\
1 & -2 & 1
\end{pmatrix},
\]

which = \( RA \). Q.E.D.

(d) Consider the vector \( y = Rx \). Using \( RA = AR \), we obtain: \( Ay = ARx = RAx = \lambda Rx = \lambda y \). Therefore, \( y \) is an eigenvector of \( A \) with eigenvalue \( \lambda \). But we were told that \( \lambda \) has multiplicity 1: this means that \( y \) must be linearly dependent on \( x \), i.e. \( y = \alpha x \) for some scalar \( \alpha \). Hence \( y = Rx = \alpha x \), and \( x \) is an eigenvector of \( R \) with eigenvalue \( \alpha \). Q.E.D
(e) (i) We just write out $Rx = e^{ikx}$:

$$
R \begin{pmatrix}
1 \\
x_2 \\
\vdots \\
x_{N-1} \\
x_N
\end{pmatrix} = 
\begin{pmatrix}
x_2 \\
x_3 \\
\vdots \\
x_N
\end{pmatrix} = e^{ikx}
\begin{pmatrix}
1 \\
x_2 \\
\vdots \\
x_{N-1} \\
x_N
\end{pmatrix}
$$

and hence $x_2 = e^{ik}$, $x_3 = e^{ik}x_2 = e^{2ik}$, and so on, or

$$
\begin{pmatrix}
x_2 \\
x_3 \\
\vdots \\
x_N
\end{pmatrix} = e^{ikx} = e^{i(N-2)k}
$$

or more simply:

$$
x_n = e^{i(n-1)k}
$$

(ii) On an eigenvector, $R^N x = e^{iNk} x = x$, and hence $e^{iNk} = 1$. This means that $Nk$ is an integer multiple of $2\pi$, i.e. $Nk = 2\pi m$ for $m = 0, 1, 2, \ldots$, giving eigenvalues

$$
\alpha_m = e^{i2\pi m/N}
$$

A little more carefully, we notice that $\alpha_N = \alpha_0$, so we have $N$ distinct eigenvalues $\frac{m = 0, 1, \ldots, N - 1}{m = 0, 1, \ldots, N - 1}$

(iii) Now that we know the eigenvectors $x_n$, we can plug it back into $Ax = \lambda x$. Each row of this equation has the form

$$
\frac{\kappa}{m} (x_{n+1} - 2x_n + x_{n-1}) = \lambda x_n
$$

and plugging in the form of $x_n = e^{i(k(n-1))} = e^{ik}e^{-ik}$ and dividing both sides by $x_n$ gives:

$$
\frac{\kappa}{m} (e^{ik} - 2 + e^{-ik}) = \lambda = \frac{\kappa}{m} [2\cos(k) - 2].
$$

Hence, plugging in the equation for $k$ from above, we have:

$$
\lambda_m = \frac{2\kappa}{m} [\cos(2\pi m/N) - 1] = -\frac{4\kappa}{m} \sin^2 \left( \frac{\pi m}{N} \right)
$$

for $m = 0, 1, \ldots, N - 1$, where we have used the half-angle identity $1 - \cos(k) = 2\sin^2(k/2)$ to simplify the final expression. Note that the eigenvalues are real and $\leq 0$ as expected, with exactly one zero eigenvalue $\lambda_0 = 0$.

(f) The angular difference between each mass is $\Delta \theta = \frac{2\pi}{N}$, and hence $x_n = e^{i\Delta \theta m(n-1)} = e^{im\theta}$ where we define the angle $\theta = (n-1)\Delta \theta$. Hence the eigenfunctions in the continuum limit are simply

$$
\phi(\theta) = e^{im\theta}
$$

for integers $m$ (or any constant multiple thereof, of course).
Problem 3: (5+5+10 points)

(a) Given the above identity, integration by parts is straightsforwards:

\[
\langle \bar{u}, \nabla \times v \rangle = \int_{\Omega} (\bar{u} \cdot (\nabla \times v)) = \int_{\Omega} [\nabla \cdot (\bar{u} \times v) + \nabla \times \bar{u} \cdot v]
\]

\[
= \iint_{\partial \Omega} (\bar{u} \times v) \cdot dS + \langle \nabla \times u, v \rangle,
\]

applying the divergence theorem in the second line. So, the surface term \(\iint_{\partial \Omega} w \cdot dS\) is for \(w = \bar{u} \times v\).

(b) If \(u \times n = 0\) on \(\partial \Omega\), then \(u\) is parallel to \(n\) and hence \(\bar{u} \times v\) is perpendicular to \(n\) and \(dS\). Hence the boundary term the integration by parts above vanishes, and \(\langle u, \nabla \times v \rangle = \langle \nabla \times u, v \rangle\), so \(\nabla \times\) is Hermitian.

(c) Taking the curl of both sides of Faraday’s Law, we have

\[
\nabla \times \nabla \times E = -\nabla \times \frac{\partial B}{\partial t} = -\frac{\partial (\nabla \times B)}{\partial t} = -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}.
\]

Under the same inner product as above, we can just “integrate by parts” twice:

\[
\langle u, \nabla \times \nabla \times v \rangle = \int_{\Omega} (\bar{u} \cdot (\nabla \times \nabla \times v)) = \iint_{\partial \Omega} [\bar{u} \times (\nabla \times \nabla \times v)] \cdot dS + \int_{\Omega} (\nabla \times \bar{u}) \cdot (\nabla \times v)
\]

\[
= \iint_{\partial \Omega} (\nabla \times \bar{u}) \cdot dS + \int_{\Omega} (\nabla \times \nabla \times \bar{u}) \cdot v = \langle \nabla \times \nabla \times u, v \rangle,
\]

where the boundary terms cancel as before under the boundary condition \(u \times n|_{\partial \Omega} = 0\). Hence \(\nabla \times \nabla \times\) will have real eigenvalues \(\lambda\). Furthermore, we can easily show that \(\nabla \times \nabla \times\) is positive semidefinite, since from above

\[
\langle u, \nabla \times \nabla \times u \rangle = \int_{\Omega} |\nabla \times u|^2 \geq 0,
\]

and hence \(\lambda \geq 0\) for some real “eigenfrequencies” \(\omega\). Equivalently, we have

\[
\hat{A}E = \frac{\partial^2 E}{\partial t^2}
\]

where \(\hat{A} = -c^2 \nabla \times \nabla \times\) is Hermitian and negative semidefinite. From class, this is a hyperbolic equation with oscillating solutions (whose frequencies \(\omega\) come from the eigenvalues \(-\omega^2\) of \(\hat{A}\))tals have high conductivity, and such containers are called microwave resonant cavities.)
Figure 1: Circular systems of $N$ identical masses $m$ and springs $\kappa$. $\phi_n$ is the angular displacement of the $n$-th mass ($\phi_m = 0$ for all springs when they are at rest). Imagine that the springs can move in the $\phi$ direction, but cannot move in the radial direction (for example, if they are sliding without friction on the surface of a cylinder of radius $R$).