18.303 Problem Set 2

Due Friday, 23 September 2016.

Problem 1: Inner products, adjoints, and definiteness

Here, we consider inner products $\langle u, v \rangle$ on some vector space $V$ of real-valued functions and the corresponding adjoint $\hat{A}^*$ of real-valued operators $\hat{A}$, where the transpose is defined, as in class, by whatever satisfies $\langle u, \hat{A}v \rangle = \langle \hat{A}^*u, v \rangle$ for all $u$ and $v$ in the vector space (usually, $\hat{A}^*$ is obtained from $\hat{A}$ by some kind of integration by parts).

(a) Suppose $V$ consists of the functions $u(x)$ on $x \in [0, L]$ with quasiperiodic boundary conditions $u(0) = e^{i\phi}u(L)$ and $u'(0) = e^{i\phi}u'(L)$, as in pset 1, and the inner product is $\langle u, v \rangle = \int_0^L u(x)v(x)dx$. Show that $\hat{A} = -d^2/dx^2$ is self-adjoint ($\hat{A} = \hat{A}^*$). For what values (if any) of $\phi$ is it positive-definite? Is this consistent with what you found in problem 3(a) of pset 1?

(b) Suppose that $\hat{A} = -d^2/dx^2 + q(x)$ where $q(x)$ is some real-valued function with $q(x) \geq q_0$ for some constant $q_0$. Show that this is self-adjoint, for the same vector space $V$ and inner product as in the first part. Furthermore, show that all eigenvalues $\lambda$ of $\hat{A}$ are $\geq q_0$. (Hint: consider whether $\hat{A} - q_0$ is definite.)

Problem 2: Modified inner products for column vectors

Consider the inner product $(x,y)_B = x^*By$ from pset 1 (problem 1b), where $B$ is a real-symmetric positive-definite matrix.

If $A$ is a real-symmetric matrix, then show that the matrix $C = B^{-1}A$ is self-adjoint with respect to the $(x,y)_B$ inner product, i.e. that $(x,Cy)_B = (Cx,y)_B$.

[ Hence the result from pset 1b (real $\lambda$ and orthogonal eigenvectors of $B^{-1}A$) follows immediately by the proof in class. ]

Problem 3: Finite-difference approximations

Suppose that we want to analyze the operation (from class)

$$\hat{A}u = c\frac{d^2u}{dx^2}$$

where $c(x) > 0$ is a real-valued positive function. Now, we want to construct a finite-difference approximation for $\hat{A}$ with $u(x)$ on $\Omega = [0, L]$ and Dirichlet boundary conditions $u(0) = u(L) = 0$, similar to class, approximating $u(m\Delta x) \approx u_m$ for $M$ equally spaced points $m = 1, 2, \ldots, M$, $u_0 = u_{M+1} = 0$, and $\Delta x = \frac{L}{M+1}$.

(a) Write down a finite-difference approximation, using center differences as in class, that corresponds to approximating $\hat{A}u$ by $Au$ where $u$ is the column vector of the $M$ points $u_m$ and $A$ is a matrix of the form $A = -CD^TD$...that is, give the matrix $C$, where $D$ is the same as the 1st-derivative matrix from lecture.

(b) Explain why you expect the matrix $A$ to have real, negative eigenvalues, even though $A \neq A^T$.

(Hint: choose the correct inner product, with help from problem 2!)

(c) In Julia, the diagm(c) command will create a diagonal matrix from a vector c. The function diagf(M) = [ 1.0 zeros(1,M-1) ; diagm(ones(M-1),1) - eye(M) ]

will allow you to create the $(M+1) \times M$ matrix $D$ from class (except missing the $1/\Delta x$ factor) via $D = \text{diff}(M)$ for any given value of $M$. Using these two commands, construct the matrix $A$ from part (a) for $M = 100$ and $L = 1$ and $c(x) = e^{3x}$ via
\( L = 1 \)
\( M = 100 \)
\( dx = L / (M+1) \)
\( D = \text{diff1}(M) / dx \)
\( x = (1:M)*dx \) # sequence of x values from dx to L-dx in steps of dx
\( C = \ldots \text{something from } c(x) \ldots \text{hint: use diagm...} \)
\( A = -C * D' * D \)

You can now get the eigenvalues and eigenvectors by
\( \lambda, U = \text{eig}(A) \), where \( \lambda \) is an array of eigenvalues and \( U \) is a matrix whose columns are the corresponding eigenvectors (notice that all the \( \lambda \) are \( < 0 \) since \( A \) is negative-definite).

(i) Plot the eigenvectors for the smallest-magnitude four eigenvalues. Since the eigenvalues are negative, by sorting them in decreasing order, these become the first four columns of \( U \). You can sort and plot them with:

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using PyPlot
i = sortperm(\( \lambda \), rev=true) # i sorts \( \lambda \) in descending order
plot(x, U[:,i[1:4]])
xlabel("x"); ylabel("eigenfunctions")
legend(["first", "second", "third", "fourth"])
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(ii) Verify that the first two eigenfunctions are indeed orthogonal for the correct inner product with \( \text{dot}(U[:,i[1]], X\cdot U[:,i[2]]) \) in Julia, where you replace \( X \) by an appropriate matrix (hint: see problem 2): the result of \( \text{dot}(\ldots) \) should be zero up to roundoff errors \( \lesssim 10^{-15} \).

(d) For \( c(x) = 1 \), we saw in class that the eigenfunctions are \( \sin(n\pi x/L) \). How do these compare to the eigenvectors you plotted in the previous part? Try changing \( c(x) \) to some other function (note: still needs to be real and \( > 0 \), and see how different you can make the eigenfunctions from \( \sin(n\pi x/L) \). Is there some feature that always remains similar, no matter how much you change \( c \)?

(e) How would the matrix \( A \) change if the boundary conditions were quasiperiodic \([u(0) = e^{i\phi}u(L)]\), as in problem 1? (Hint: review how we derived the \( A \) or \( D \) matrices in class, and look at the first and last rows.)