November 9, 2016

Problem 1: Hermitian (33 points)

We just choose the inner product
\[ \langle \mathbf{F}, \mathbf{G} \rangle \varepsilon = \int \mathbf{F} \cdot \varepsilon \mathbf{G} \]
in order to cancel the \( \varepsilon^{-1} \) factor in \( \hat{A} \). (Note that this is a valid inner product since \( \varepsilon > 0 \), much like the weighted inner product we used for \( c \nabla^2 \) in class.) Then
\[
\langle \mathbf{E}, \hat{A} \mathbf{E}' \rangle \varepsilon = \langle \mathbf{E}, \nabla \times \mu^{-1} \varepsilon \nabla \times \mathbf{E}' \rangle
= \langle \nabla \times \mathbf{E}, \mu^{-1} \nabla \times \mathbf{E}' \rangle
= \langle \mathbf{E}, \varepsilon^{-1} \nabla \times \mu^{-1} \nabla \times \mathbf{E}' \rangle
= \langle \hat{A} \mathbf{E}, \mathbf{E}' \rangle \varepsilon,
\]
where we have integrated by parts twice with \( \nabla \times \), using the identity from homework, assuming we have boundary conditions such that the boundary terms vanish as in homework. Hence \( \hat{A} = \hat{A}^* \). To check definiteness, we just look at the “middle” step from the end of the first line to see that
\[
\langle \mathbf{E}, \hat{A} \mathbf{E} \rangle \varepsilon = \langle \nabla \times \mathbf{E}, \mu^{-1} \nabla \times \mathbf{E} \rangle = \int_{\Omega} \mu^{-1} |\nabla \times \mathbf{E}|^2 \geq 0
\]
since \( \mu > 0 \). Hence \( \hat{A} = \hat{A}^* \succeq 0 \) and we will obtain oscillating solutions for \( \partial^2 \mathbf{E} / \partial t^2 = \hat{A} \mathbf{E} \).

A common mistake in this problem was to choose an inner product \( \langle \mathbf{F}, \mathbf{G} \rangle = \int \varepsilon \mu \bar{\mathbf{F}} \cdot \mathbf{G} \), and then to claim that \( \langle \mathbf{E}, \hat{A} \mathbf{E}' \rangle = \int \varepsilon \mu \bar{\mathbf{E}} \cdot \nabla \times \mu^{-1} \nabla \times \mathbf{E}' = \int \bar{\mathbf{E}} \cdot \nabla \times \mathbf{E}' \), which is not true since \( \mu(x) \) is not a constant (you can’t interchange it with \( \nabla \times \)).

Problem 2: Timestepping (34 points)

1. We use the Taylor series around \( n + \frac{1}{2} \):
\[
\mathbf{u}^{n+\frac{1}{2}+\frac{1}{2}} = \mathbf{u}(n + \frac{1}{2}\Delta t) \pm \Delta t \dot{\mathbf{u}}(n + \frac{1}{2}\Delta t) + O(\Delta t^2),
\]
where + gives \( \mathbf{u}^{n+1} \) and − gives \( \mathbf{u}^{n} \). Then
\[
\frac{\mathbf{u}^{n+1} + \mathbf{u}^{n}}{2} = \mathbf{u}(n + \frac{1}{2}\Delta t) + \Delta t \dot{\mathbf{u}}(n + \frac{1}{2}\Delta t) + O(\Delta t^2) + \mathbf{u}(n + \frac{1}{2}\Delta t) - \Delta t \dot{\mathbf{u}}(n + \frac{1}{2}\Delta t) + O(\Delta t^2)
= \mathbf{u}(n + \frac{1}{2}\Delta t) + O(\Delta t^2),
\]
as desired.

It is also possible to do this by Taylor-expanding around \( \mathbf{u}(n\Delta t) \), and comparing the result to the Taylor series of \( \mathbf{u}(n + \frac{1}{2}\Delta t) \):
\[
\frac{\mathbf{u}^{n+1} + \mathbf{u}^{n}}{2} = \frac{\mathbf{u}(n\Delta t) + \Delta t \dot{\mathbf{u}}(n\Delta t) + O(\Delta t^2) + \mathbf{u}(n\Delta t)}{2} = \mathbf{u}(n\Delta t) + \frac{\Delta t}{2} \dot{\mathbf{u}}(n\Delta t) + O(\Delta t^2) = \mathbf{u}(n + \frac{1}{2}\Delta t) + \mathbf{u}(n + \frac{1}{2}\Delta t).
\]
2. Solving for \( u^{n+1} \), we have

\[
\begin{align*}
    u^{n+1} &= \left( I - \frac{A\Delta t}{2} \right)^{-1} \left( I + \frac{A\Delta t}{2} \right) u^n = Bu^n,
\end{align*}
\]

where we have defined the matrix \( B \), and hence

\[
    u^n = B^n u^0
\]

as in class. If \( \lambda \) is an eigenvalue of \( A \) for some eigenvector, then the same vector is an eigenvector of \( B \) with eigenvalue \( \mu = (1 + \lambda\Delta t/2)/(1 - \lambda\Delta t/2) \). If \( A = A^* < 0 \), then \( \lambda < 0 \), and it follows that \( |\mu| < 1 \) for any \( \Delta t > 0 \) (the denominator of \( \mu \) is bigger than the numerator, since addition gives a bigger number than subtraction). Hence \( B^n \to 0 \) as \( n \to \infty \), and the scheme is unconditionally stable.

By the way, a common mistake here is to write \( u^{n+1} = I + A\Delta t/2 \), which is “not even wrong!” if \( B \) and \( C \) are matrices, the expression \( B^{-1}C \) is meaningless because it is not clear whether you mean \( B^{-1}C \) or \( C^{-1}B \), unless they happen to commute, which they don’t in this case). Another common mistake is to check that \( \mu < 1 \), which is not sufficient: you need \( |\mu| < 1 \) for the solutions to decay.

**Problem 3: Born (33 points)**

We write

\[
\hat{A}(\Delta p) = -\nabla^2 + c(\Delta p, x) = \hat{A}(0) + \frac{\partial c}{\partial p} \bigg|_{p=0} \Delta p + O(\Delta p^2)
\]

by Taylor-expanding \( c \) around \( p = 0 \). Then, by moving the \( \partial c/\partial p \) term to the right-hand-side, we see that \( \hat{A}(\Delta p)u = f \) solves

\[
    u = \hat{A}(0)^{-1} \left[ f - \frac{\partial c}{\partial p} \bigg|_{p=0} u \Delta p + O(\Delta p^2) \right].
\]

Now, plugging in the right-hand-side for \( u \), as in the derivation of the Born–Dyson series in class, we obtain

\[
    u = \hat{A}(0)^{-1} \left[ f - \frac{\partial c}{\partial p} \bigg|_{p=0} \hat{A}(0)^{-1} f \Delta p + O(\Delta p^2) \right],
\]

where we have lumped all terms of order \( \Delta p^2 \) or higher together, and the second term is the first Born approximation. Now, to get the derivative, we do

\[
    \frac{\partial u}{\partial p} \bigg|_{p=0} = \lim_{\Delta p \to 0} \frac{u|_{p=\Delta p} - u|_{p=0}}{\Delta p} = \lim_{\Delta p \to 0} \frac{\hat{A}(0)^{-1} \left[ f - \frac{\partial c}{\partial p} \bigg|_{p=0} \hat{A}(0)^{-1} f \Delta p + O(\Delta p^2) \right] - \hat{A}(0)^{-1} f}{\Delta p}.
\]

In fact, one can easily generalize this approach to show that, for any invertible operator that depends in a differentiable way on a parameter \( p \), the derivative of the inverse of the operator is:

\[
    \frac{\partial}{\partial p} \hat{A}^{-1} = -\hat{A}^{-1} \frac{\partial \hat{A}}{\partial p} \hat{A}^{-1},
\]

which is a generalization of the chain rule \( \frac{\partial}{\partial p} a(p)^{-1} = -\frac{\partial a/\partial p}{a^2} \) from first-year calculus.