Problem 1: Hermitian (33 points)

In homework, you showed that \( \nabla \times \) was Hermitian under the inner product \( \langle F, G \rangle = \int \overline{F} \cdot G \); that is, \( \int \nabla \times \overline{F} \cdot G = \int F \cdot \nabla \times G \), for appropriate boundary conditions, and from that you went on to show that \( \nabla \times \nabla \times \) was Hermitian and positive-semidefinite, and hence Maxwell’s equations \( \frac{\partial^2 E}{\partial t^2} = \nabla \times \nabla \times E \) had oscillating solutions.

The case you analyzed in homework only applied to Maxwell’s equations in vacuum, however. In materials, the equations become

\[
\frac{\partial^2 E}{\partial t^2} = \varepsilon^{-1} \nabla \times (\mu^{-1} \nabla \times E),
\]

where \( \mu(x) \) and \( \varepsilon(x) \) are material properties related to the magnetic and electric polarizability of matter.

Assuming \( \mu \) and \( \varepsilon \) are real, positive scalar functions, choose an inner product and show that the operator \( \hat{A} = \varepsilon^{-1} \nabla \times (\mu^{-1} \nabla \times) \) is Hermitian and positive semidefinite, and hence that we still have oscillating solutions. Don’t worry about the boundary conditions—just assume that we have chosen boundary conditions so that \( \nabla \times \) is still Hermitian as above (i.e. so that boundary terms vanish when you integrate by parts, i.e. you can just use the integral identity from above).

Problem 2: Timestepping (34 points)

Suppose we have a PDE \( \frac{\partial u}{\partial t} = \hat{A} u \), that we discretize via finite-differences as in class into a “Crank-Nicolson” scheme:

\[
\frac{u^{n+1} - u^n}{\Delta t} = A \frac{u^{n+1} + u^n}{2},
\]

where \( u^n \) denotes the (discretized) \( u(x) \) at time \( t = n\Delta t \), and \( A \) is a discretized version of the operator \( \hat{A} \) (e.g. finite differences in space if \( \hat{A} \) consists of spatial derivatives like \( \nabla^2 \)).

1. From class, the left-hand side is a second-order accurate (errors \( \sim \Delta t^2 \)) center-difference approximation for \( \frac{\partial u}{\partial t} \) at time \( t = (n + \frac{1}{2})\Delta t \). That requires the right-hand side to also be at time \( n + \frac{1}{2} \). If we treat \( u^n \) as an approximation for \( u(n\Delta t) \), show that \( \frac{u^{n+1} - u^n}{2} \approx u([n + \frac{1}{2}]\Delta t) + O(\Delta t^2) \), i.e. it is second-order accurate. (This was claimed in class but not proved. Use the Taylor series.)

2. Suppose our matrix \( A \) (independent of \( n \)) satisfies \( A = A^* \prec 0 \) (negative definite). Show that the solutions \( u^n \) of our finite-difference scheme above go to zero as \( n \to \infty \). (Hint: write \( u^n = \text{(something)}^n u^0 \) as in class and write the eigenvalues of the “something” in terms of the eigenvalues of \( A \) i.e. show it is unconditionally stable (for any \( \Delta t > 0 \)).

Problem 3: Born (33 points)

Consider the operator \( \hat{A}(p) = -\nabla^2 + c(p, x) \) in some domain \( \Omega \) with Dirichlet boundary conditions \( u|_{\partial \Omega} = 0 \), where \( c(p, x) \) is a real-valued function that depends on some parameter \( p \) (\( p \) is a real number). Suppose that I tell you that I have a computer program that can quickly and accurately solve

\[
\hat{A}(0)u = f
\]
for \( u(\mathbf{x}) \) given any right-hand side \( f(\mathbf{x}) \). That is, I can apply \( \hat{A}(0)^{-1} \).

Explain how, using my \( \hat{A}(0)^{-1} \) computer program (without modifying it), you can compute (for a given \( f \))

\[
\left. \frac{\partial u}{\partial p} \right|_{p=0} .
\]

Hint: write the solution for a small \( \Delta p \) approximately using a Born approximation, then take the \( \Delta p \to 0 \) limit. Assume you can Taylor-expand \( c(p, \mathbf{x}) \) around \( p = 0 \).