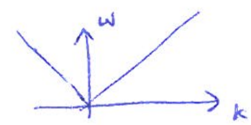
Phase velocity, Group velocity + Fourier transforms

* The simplest solutions to wave equations (for constant coefs) are plane waves $u(x,t) = e^{i(k\cdot x - \omega t)}$

where $\omega(k)$ is the dispersion relation

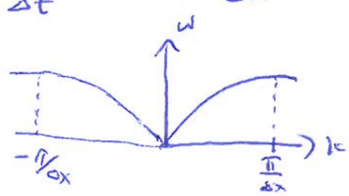
examples:

- $\omega = \pm c|k|$ for $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

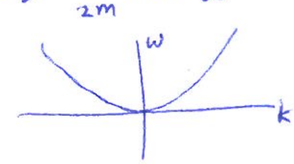


- $\omega = \pm \frac{2}{\Delta t} \sin^{-1}\left(\frac{c\Delta t}{\Delta x} \sin\left(\frac{k\Delta x}{2}\right)\right)$ for center-difference:

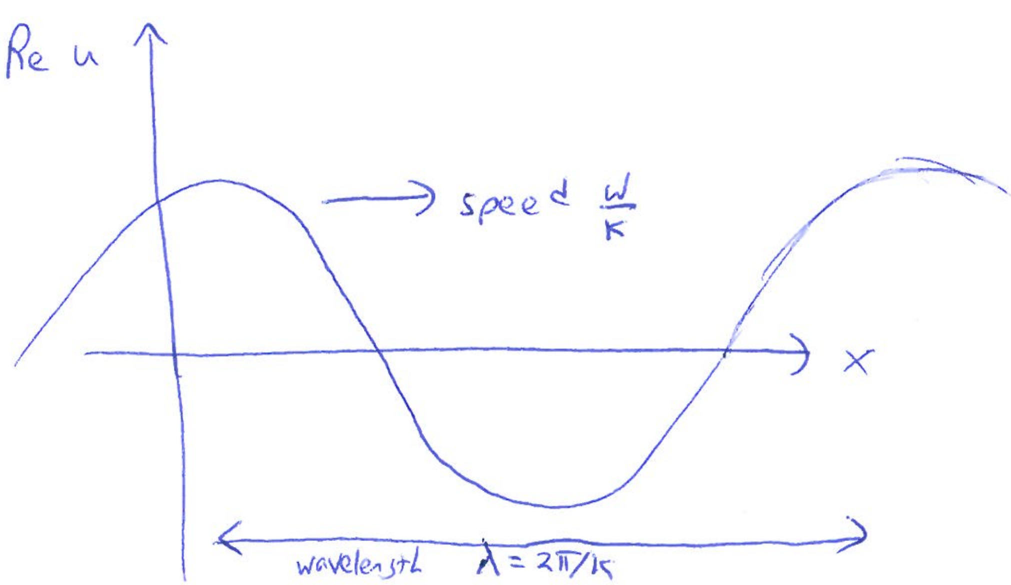
$$c^2 \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{\Delta x^2} = \frac{u^{n+1}_m - 2u^n_m + u^{n-1}_m}{\Delta t^2}$$



- for 1d Schrödinger equation: $-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial x^2} = i\hbar \frac{\partial u}{\partial t} \Rightarrow \frac{\hbar}{2m} k^2 = \omega$



* By inspection, $u = e^{i(kx - \omega t)} = e^{ik\left(x - \frac{\omega}{k} t\right)}$

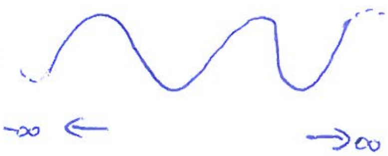


$\Rightarrow$ phase velocity $= v_p = \frac{\omega}{k}$

= speed of "ripples"

* Is v_p a "useful" velocity?

- a planewave is infinitely extended in space

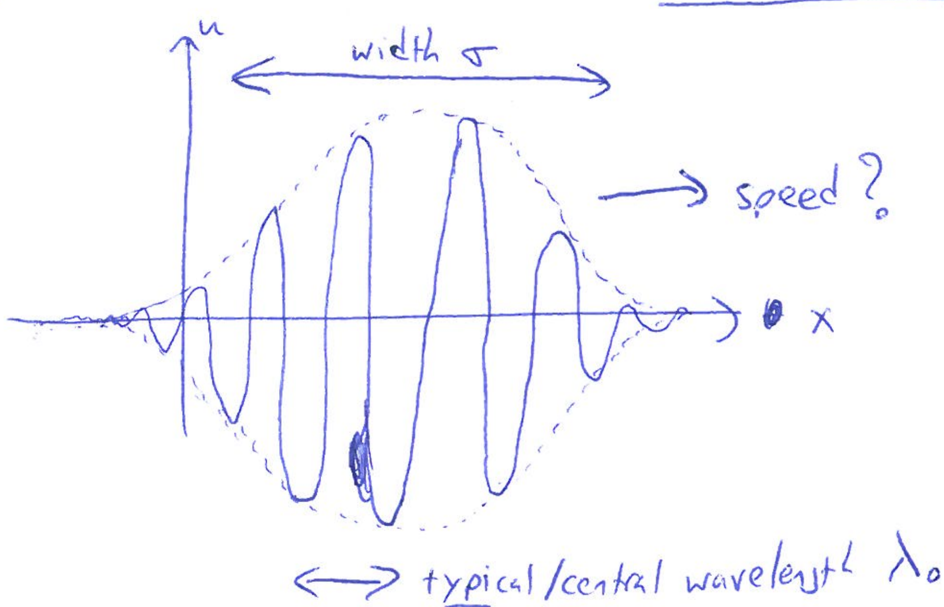


$\Rightarrow$ can never be said to "leave" or "arrive" anywhere

$\Rightarrow$ traditional understanding of velocity as "travel time" is questionable

- i.e. planewaves, by themselves, cannot transmit information

* Instead, we want to consider a wave packet ("pulse")



- to understand the speed at which a wavepacket travels (can truly "leave"/"arrive"/carry info.) we need to write it as a

superposition of planewaves = Fourier transform

Fourier transforms:

$f(x) \longleftrightarrow \hat{f}(k)$ amplitude of e^{ikx}

space (or time) domain — frequency domain

So far: ① Fourier series: periodic $f(x) = \frac{\sqrt{2\pi}}{L}\sum_{n=-\infty}^{\infty} \hat{f}_n e^{i\frac{2\pi}{L}nx}$ on $[-\frac{L}{2}, \frac{L}{2}]$

(renormalizing in a more symmetrical $1/\sqrt{2\pi}$ way)

$k_n = \frac{2\pi}{L}n$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_n \hat{f}_n e^{ik_n x} \underbrace{\Delta k}_{= \frac{2\pi}{L}}$$

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{+L/2} f(x) e^{-ik_n x} dx$$

② DTFT discrete time/space Fourier transform:

discrete $f(m\Delta x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\Delta x}^{+\pi/\Delta x} \hat{f}(k) e^{ikm\Delta x} dk$

(Fourier series in "reverse", $L = \frac{\pi}{\Delta x}$)

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} f(m\Delta x) e^{-ikm\Delta x} \cdot \Delta x$$

$*$ $\lim_{L\to\infty}$ ① or $\lim_{\Delta x \to 0}$ ②

$\Rightarrow$ Fourier transform

"any" — tempered distribution (at most polynomially growing with x)

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk}$$

$$\boxed{\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx}$$

periodic $f(x)$ $\Longleftrightarrow$ $\hat{f}(k) = \sum_n \hat{f}_n \delta(k - k_n)$

discrete $f(x) = \sum_m f(m\Delta x)\, \delta(x - m\Delta x)$

* A few important properties (out of many)

- $\hat{f}(k) = \delta(k-k_0) \underset{F.T.}{\Longleftrightarrow} f(x) = \frac{1}{\sqrt{2\pi}} e^{ik_0x}$

$$\Rightarrow \delta(k-k_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-i(k-k_0)x} dx$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{\pm i(k-k_0)x} dx = 2\pi\, \delta(k-k_0)$$

- $f'(x) \underset{F.T.}{\longleftrightarrow} ik\,\hat{f}(k)$

$f''(x) \Longleftrightarrow -k^2 \hat{f}(k)$

$\vdots$

- $e^{-ikx_0}\hat{f}(k) \underset{F.T.}{\Longleftrightarrow} f(x-x_0)$ (also: $f(x)e^{ik_0x} \Leftrightarrow \hat{f}(k-k_0)$)

- $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$ unitarity / Parsevals theorem / Plancherel's theorem

(pf) $$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} dx \overline{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \hat{f}(k) e^{ikx}\right]} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk'\, \hat{f}(k') e^{ik'x}\right]$$

$$= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'\, \overline{\hat{f}(k)}\, \hat{f}(k') \underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(k-k')x} dx\right]}_{=\delta(k-k')} = \int_{-\infty}^{\infty} dk\, |\hat{f}(k)|^2$$

* "Uncertainty principle":

(loosely) the more "localized" $f(x)$ is in space, the less "localized" $\hat{f}(k)$ is in frequency, + vice versa

ex: $f(x) = \delta(x - x_0)$ (localized at one point x_0)

$$\Longleftrightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0}$$

($|\hat{f}| = 1/\sqrt{2\pi}$ for all k)

ex: $f(x) = \begin{cases} e^{ik_0 x} & |x| < \frac{w}{2} \\ 0 & |x| \geq \frac{w}{2} \end{cases}$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

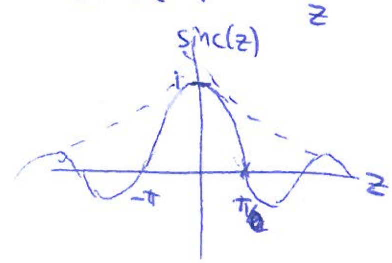
$$= \frac{1}{\sqrt{2\pi}} \int_{-w/2}^{+w/2} e^{-i(k-k_0)x} dx$$

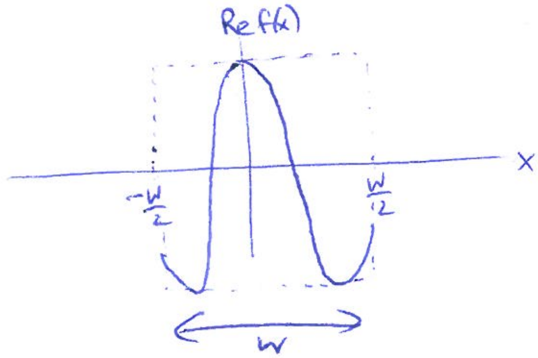
$$= \frac{e^{+i(k-k_0)w/2} - e^{-i(k-k_0)w/2}}{\sqrt{2\pi}\, i\,(k-k_0)}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin\left[(k-k_0)\frac{w}{2}\right]}{(k-k_0)}$$

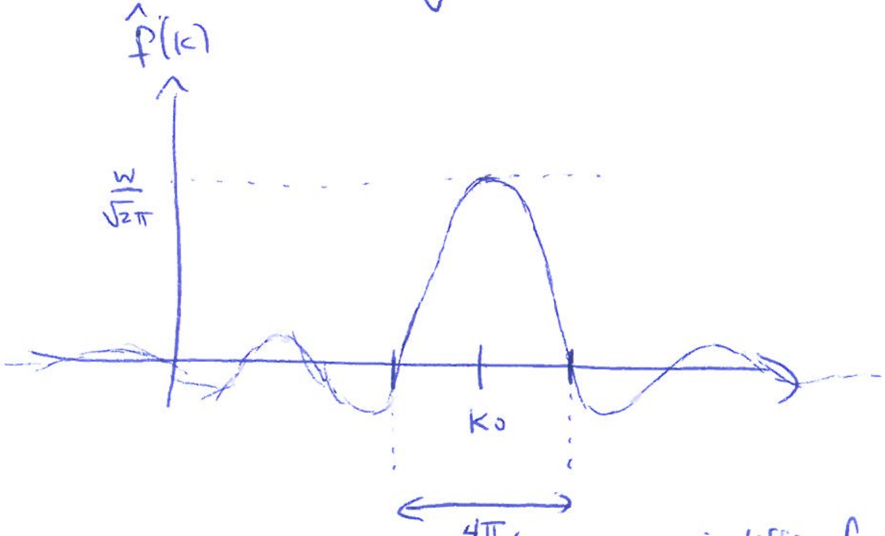
$$= \frac{w}{\sqrt{2\pi}} \operatorname{sinc}\left[(k-k_0)\frac{w}{2}\right]$$

$$\operatorname{sinc}(z) = \frac{\sin(z)}{z}$$





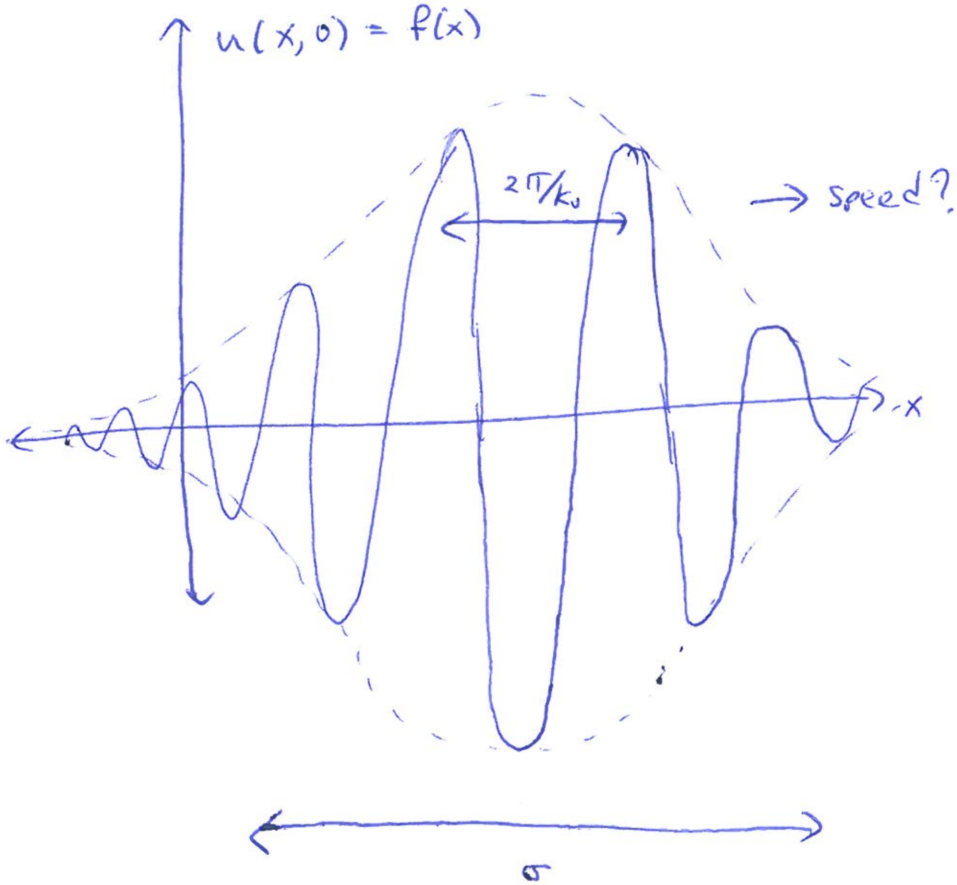
F.T.

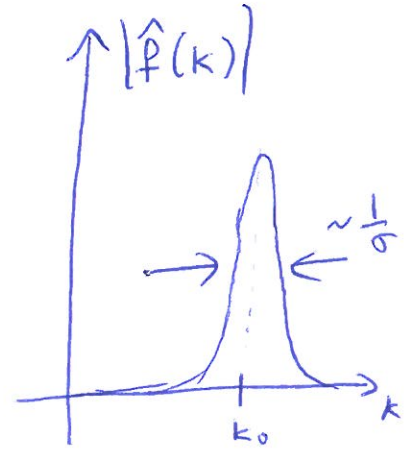


~ inverse of f width

Group velocity :

consider a wavepacket wide in x, narrow in k :





suppose all Fourier components have $v_p = \frac{\omega}{k} > 0$:

$$\Rightarrow \text{solution} \quad u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) \underbrace{e^{i[kx - \overset{\text{some dispersion relation}}{\omega(k)}\, t]}}_{\substack{\text{superposition of} \\ \text{planewaves moving} \rightarrow}} dk$$

$*$ key point: since $|\hat{f}(k)| \approx 0$ except near k_0,
we only need to know $\omega(k)$ near k_0

$$\Longrightarrow \text{Taylor expand : } \omega(k) \approx \omega(k_0) + \omega'(k_0)(k-k_0) + \ldots$$

$$\Rightarrow u(x,t) \approx \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ik[x-\omega'(k_0)t]}\, dk \right) \cdot \underbrace{e^{i[\omega(k_0)-\omega'(k_0)k_0]t}}_{k\text{-independent}}$$

$$= f(x-\omega'(k_0)t) \cdot e^{i[\omega(k_0)-\omega'(k_0)k_0]t}$$

$$= \left(\begin{array}{l} \text{initial envelope/wavepacket} \\ \text{moving at} \\ \text{speed:} \end{array} \right) \cdot \left(\begin{array}{l} \text{ripples / phase oscillations} \\ * \text{ ripples move at} \sim v_p = \frac{\omega}{k}\Big|_{k_0} \neq v_g \end{array} \right)$$

$$\boxed{v_g = \frac{d\omega}{dk}\Big|_{k_0} = \text{group velocity}}$$

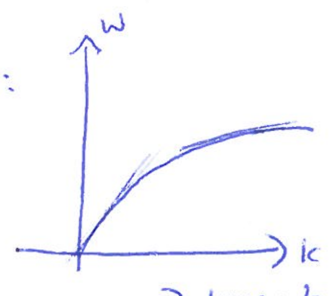
Group velocity dispersion:

$\frac{d\omega}{dk}$ depends (in general) on k (or ω)

$\Longrightarrow$ wave packets spread out ("disperse")

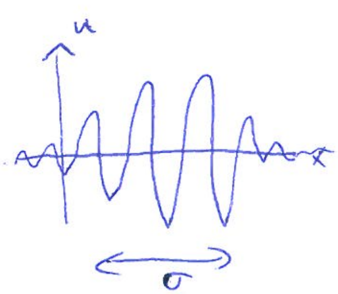
— slower k components behind, faster k components in front

ex:

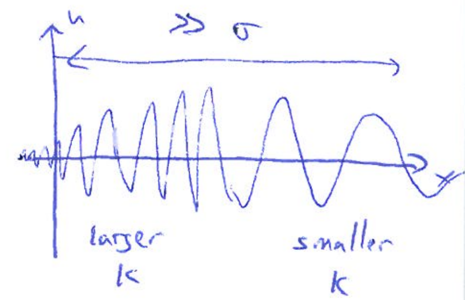


→ larger k = slower

$\Longrightarrow$



long distance →



("chirped" pulse)

quantifying dispersion:

- consider pulse duration $T = \sigma / v_g$ (width in time)
- pulse contains some range of k's : $\Delta k \sim \frac{1}{\sigma}$

$=$ range of ω's $\Delta\omega \sim \Delta k \cdot \frac{d\omega}{dk} = v_g \Delta k = \frac{v_g}{\sigma} = 1/T$

$=$ range of group velocities

after a distance $L \gg \sigma$,

width in time $$\Delta t \approx \underbrace{\frac{L}{v_{min}}}_{\text{slowest } v_g \text{ k's}} - \underbrace{\frac{L}{v_{max}}}_{\text{k's with fastest } v_g} = L \, \Delta\left(\frac{1}{v}\right)$$

$$\approx L \overbrace{\frac{d\left(\frac{1}{v_g}\right)}{d\omega}}^{\frac{dk}{d\omega} \,\,''} \Delta\omega = L \frac{d^2k}{d\omega^2} \frac{1}{T}$$

$\Longrightarrow$ spreads ~ linearly with $\underline{\underline{L}}$, $\underline{\underline{\frac{d^2k}{d\omega^2}}}$ $\left(\neq \left(\frac{d^2\omega}{dk^2}\right)^{-1}!\right)$, $\underbrace{\frac{1}{T}}_{\text{bandwidth}}$

Where does dispersion come from?

* in $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, solution $e^{i(kx-\omega t)}$ for $\omega = ck \Rightarrow \frac{d\omega}{dk} = \frac{\omega}{k} = c$
$= \text{constant}$
(no dispersion)

here, equation is scale-invariant : let $\tilde{x} = sx$, $\tilde{t} = st$ $\Rightarrow$ same equation $c^2 \frac{\partial^2 u}{\partial \tilde{x}^2} = \frac{\partial^2 u}{\partial \tilde{t}^2}$

$\Longrightarrow$ solution + speed cannot depend on scale (e.g. wavelength $\left(\frac{2\pi}{k}\right)$ or frequency ω)

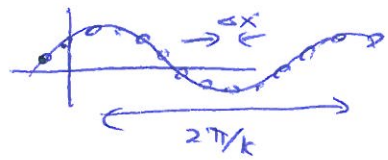
* Dispersion arises when the system / solution responds differently at different spatial or time scales

Sources of dispersion:

1) Numerical dispersion: discretization of space/time sets Δx + Δt length/time scales

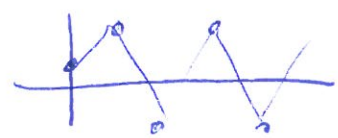
- solution is very different for

$k\Delta x \ll 1$



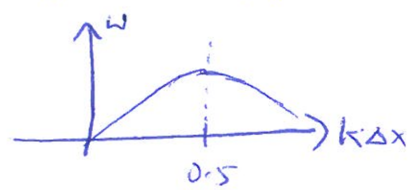
$\approx$ continuous equation

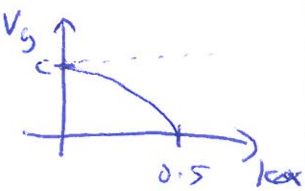
$k\Delta x \gtrsim 1$



very discrete (very different from contin.)

$\Longrightarrow$ speed depends strongly on $k\Delta x$ (or $w\Delta t$)





2) Material dispersion

real materials respond differently at different w $\xleftrightarrow[\text{(convolution theorem)}]{\text{Fourier}}$ real materials don't respond instantaneously to stimuli

c depends on w

ex: index of refraction (optics) depends on w $\longleftrightarrow$ matter does not polarize instantly in response to $\vec{E}$ fields

$\Rightarrow$ speed = c/index depends on w

$\Longrightarrow$ rainbows!

convolutions, dispersion, & instantaneity:

- consider solutions in frequency domain $e^{-i\omega t} \cdot \hat{u}(x,\omega)$

to scalar wave equation: $c^2 \frac{\partial^2 \hat{u}}{\partial x^2} = -\omega^2 \hat{u}$

+ suppose $c(\omega)$ depends on ω (material dispersion)

... what does equation look like in time domain?

let $\hat{\chi}(\omega) = c^2(\omega)$:

"susceptibility"

$$\underbrace{\hat{\chi}(\omega) \frac{\partial^2 \hat{u}}{\partial x^2}}_{\text{product in } \omega \text{ domain}} = -\omega^2 \hat{u}$$

Fourier: $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(x,\omega) e^{-i\omega t} d\omega$

$$\underbrace{\chi(t) * \frac{\partial^2 u}{\partial x^2}}_{\text{convolution in } \omega \text{ domain}} = \frac{\partial^2 u}{\partial t^2}$$

= non-instantaneous response ($\frac{\partial^2 u}{\partial t^2}$ depends on $\frac{\partial^2 u}{\partial x^2}$ in the past)

explicitly:

$$\frac{\partial^2 u}{\partial t^2}\Big|_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\chi}(\omega) \frac{\partial^2 \hat{u}}{\partial x^2} e^{-i\omega t} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(t') e^{i\omega t'} dt'\right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\Big|_{t''} e^{i\omega t''} dt''\right] e^{-i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \, \chi(t') \frac{\partial^2 u}{\partial x^2}\Big|_{t''} \underbrace{2\pi \int_{-\infty}^{\infty} d\omega \, e^{i\omega(t'+t''-t)} d\omega}_{= \delta(t'+t''-t)}$$

$$= \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(t-t'') \frac{\partial^2 u}{\partial x^2}\Big|_{t''} dt'' = \chi * \frac{\partial^2 u}{\partial x^2}}$$

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_t = \chi * \left.\frac{\partial^2 u}{\partial x^2}\right|_t = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\chi(t-t') \left.\frac{\partial^2 u}{\partial x^2}\right|_{t'} dt'}_{\text{non-instantaneous response}}$$

<u>causality</u>: $\frac{\partial^2 u}{\partial t^2}$ can only depend on $\frac{\partial^2 u}{\partial x^2}$ in <u>past</u> $(t' \leq t)$

<u>not</u> the future $(t' > t)$

$$\Longrightarrow \chi(t-t') = 0 \quad \text{for } t' > t$$

$$\Longrightarrow \underline{\underline{\chi(\tau) = 0}} \quad \text{for } \tau < 0$$

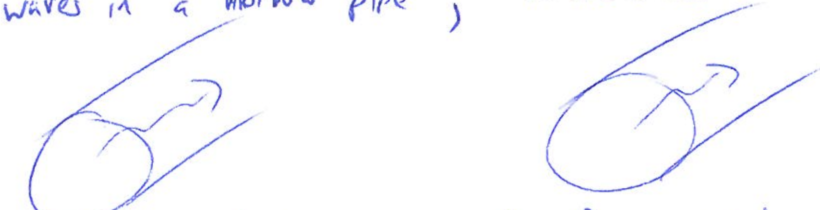
(+ complex analysis $\Longrightarrow$ lots of constraints on $\hat{\chi}(\omega)$
(Kramers-Kronig relations)
e.g. $\hat{\chi}$ is generally <u>complex</u>
$\Longleftrightarrow$ dissipation loss!)

3) <u>Waveguide / geometric dispersion</u>

= waves propagate in some inhomogeneous <u>geometry</u>
that sets a <u>lengthscale</u> $\Longrightarrow$ dispersion

ex: waves in a "pipe":

sound waves in a hollow pipe, microwaves in a metal tube



$\Longrightarrow$ very different solutions + speeds for wavelength $\gg$ diameter
or $\ll$ diameter!