1. (7.1.30 in Colley, 4th edition) Let $S$ be the surface defined by 

$$z = \frac{1}{\sqrt{x^2 + y^2}} \text{ for } z \geq 1.$$

(a) Sketch the graph of this surface.

(b) Show that the volume of the region bounded by $S$ and the plane $z = 1$ is finite. (You will need to use an improper integral.)

(c) Show that the surface area of $S$ is infinite.

Solution. (a) See the graph below.

(b) The volume is given by 

$$\int_0^1 \int_0^{2\pi} (r^2 - 1) r d\theta dr = \int_0^1 \int_0^{2\pi} (1 - r) d\theta dr,$$

which is finite since the integrand is bounded and the region of integration is compact. (c) The surface area of $S$ is given by

$$\int_0^1 \int_0^{2\pi} \sqrt{1 + f_x^2 + f_y^2} r d\theta dr = \int_0^1 \int_0^{2\pi} \sqrt{1 + x^2/(x^2 + y^2)^3 + y^2/(x^2 + y^2)^3} r d\theta dr$$

$$= \int_0^1 \int_0^{2\pi} \sqrt{1 + r^{-4}} r d\theta dr$$

$$= 2\pi \int_0^1 \sqrt{r^2 + r^{-2}} dr.$$ 

This integral is infinite because the second term dominates, and $\int_0^1 \frac{dr}{r} = +\infty$. To prove this rigorously, we can drop the first term:

$$2\pi \int_0^1 \sqrt{r^2 + r^{-2}} dr \geq 2\pi \int_0^1 \sqrt{r^{-2}} dr = +\infty. \quad \Box$$

2. (7.2.1 in Colley, 4th edition) Let $X(s, t) = (s, s + t, t), 0 \leq s \leq 1, 0 \leq t \leq 2$. Find $\iint_X (x^2 + y^2 + z^2) dS$. 


Solution. We have

\[
\int_{0}^{1} \int_{0}^{2} \left( s^2 + (s + t)^2 + t^2 \right) \sqrt{3} \sqrt{\left( \frac{\partial(x, y)}{\partial(s, t)} \right)^2 + \left( \frac{\partial(x, z)}{\partial(s, t)} \right)^2 + \left( \frac{\partial(y, z)}{\partial(s, t)} \right)^2} \ dt \ ds = \frac{26 \sqrt{3}}{3} \quad \square
\]

3. (7.2.27 in Colley, 4th edition) Let \( S \) be the funnel-shaped surface defined by \( x^2 + y^2 = z^2 \) for \( 1 \leq z \leq 9 \) and \( x^2 + y^2 = 1 \) for \( 0 \leq z \leq 1 \).

(a) Sketch \( S \).

(b) Determine outward-pointing unit normal vectors to \( S \).

(c) Evaluate \( \iint_{S} F \cdot dS \), where \( F = -yi + xj + zk \) and \( S \) is oriented by outward normals.

Solution. (a) See the graph below.

(b) The outward pointing unit vectors on the cylindrical part of the cylinder are \( xi + yj \). On the lateral face, the unit normal is \( \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -\frac{1}{\sqrt{x^2 + y^2}} \right) \).

(c) We calculate over the cylindrical surface \( S_1 \)

\[
\iint_{S_1} F \cdot dS = \iint_{S_1} (-y, x, z) \cdot (x, y, 0) \ dS = \iint_{S_1} 0 \ dS = 0.
\]

Over the conical surface, we calculate

\[
\iint_{S_2} F \cdot dS = \iint_{S_1} (-y, x, z) \cdot \left( \frac{x}{z \sqrt{2}}, \frac{y}{z \sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \ dS
\]

\[
= -\frac{1}{\sqrt{2}} \iint_{S_1} z \ dS.
\]
To evaluate this surface integral, we use slices perpendicular to the \( z \)-axis. The slice at height \( z \) is a thin circular strip of radius \( z \) and width \( z \sqrt{2} \) (the factor of \( \sqrt{2} \) arising from the 45° lean). Thus

\[
- \frac{1}{\sqrt{2}} \int_S z \, dS = - \frac{1}{\sqrt{2}} \int_1^9 2\pi z (z \sqrt{2}) \, dz
\]

\[
= 2\pi \left[ \frac{z^3}{3} \right]_1^9
\]

\[
= -1456\pi/3.
\]

Adding the contributions from \( S_1 \) and \( S_2 \), we get \(-1456\pi/3\). □