1. Find the second order Taylor polynomial for \( f(x, y) = \cos(x + 2y) \) at the origin. What is the second order Taylor polynomial for \( g(\theta) = \cos \theta \) at \( \theta = 0 \)?

**Solution.** The second order Taylor polynomial is
\[
\begin{align*}
&f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + f_{xy}(0, 0)xy + \frac{1}{2} f_{xx}(0, 0)x^2 + \frac{1}{2} f_{yy}(0, 0)y^2,
\end{align*}
\]
which equals
\[
1 - 2xy - \frac{1}{2}x^2 + 2y^2.
\]
This can also be obtained by substituting \( x + 2y \) into the Taylor polynomial \( 1 - \frac{1}{2} \theta^2 \) for \( \cos \theta \) at \( \theta = 0 \).

2. (a) Find the critical points of \( f(x, y) = x^2 + 4xy + y^2 \). Use the second derivative test for local extrema to determine whether the point is a local maximum, a local minimum, or a saddle point.

(b) Find the critical points of \( g(x, y) = x^2 + xy + y^2 \). Use the second derivative test for local extrema to determine whether the point is a local maximum, a local minimum, or a saddle point.

**Solution.** (a) The gradient of \( f \) is \((2x + 4y, 4x + 2y)\), which equals \( 0 \) if and only if \( (x, y) = (0, 0) \). Therefore, the origin is the only critical point of \( f \). The Hessian evaluated at \( (0, 0) \) is
\[
\begin{vmatrix}
2 & 4 \\
4 & 2
\end{vmatrix} = 2 \cdot 2 - 4 \cdot 4 < 0,
\]
so the origin is a saddle point.

(b) The gradient of \( g \) is \((2x + y, x + 2y)\), which equals \( 0 \) if and only if \( (x, y) = (0, 0) \). Therefore, the origin is the only critical point of \( g \). The Hessian evaluated at \( (0, 0) \) is
\[
\begin{vmatrix}
2 & 1 \\
1 & 2
\end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 > 0,
\]
so the critical point is a local extremum. Since \( f_{xx} > 0 \), the Hessian is positive definite and the critical point is a local minimum.

3. (a) What theorem ensures that the function \( f(x, y) = x \sin(x + y) \) defined on the rectangle \( \{(x, y) : 0 \leq x \leq \pi, 0 \leq y \leq 7\} \) has an absolute maximum and an absolute minimum? Verify the hypotheses of that theorem.

(b) Find the absolute extrema of \( f \). You are given that there are no absolute extrema on the top or bottom of the rectangle; see the surface plot below to guide your intuition.

**Solution.** (a) The extreme value theorem ensures that the function achieves absolute extrema, because it is continuous function defined on a compact (that is, closed and bounded) set.
(b) If \( f \) has an extremum in the interior of the rectangle, then \( Df = 0 \) there. Since \( Df = (\sin(x + y) + x\cos(x + y), x\cos(x + y)) \), there are no critical points in the interior of the rectangle. To see this, note that the second coordinate is zero if and only if \( \cos(x + y) = 0 \). If \( \cos(x + y) = 0 \), then the first coordinate is zero if and only if \( \sin(x + y) = 0 \). But sine and cosine never vanish simultaneously, so there are no critical points.

It follows that \( f \) has its absolute extremum on the edges or at one of the vertices of the rectangle. We look at each side one at a time.

- On the bottom side of the rectangle, \( f(x, y) = f(x, 0) = x\sin x \), which has a minimum of 0 at \((0, 0)\) and \((\pi, 0)\) and a maximum of about 1.81 at about \((2.02, 0)\).
- On the top side of the rectangle, \( f(x, y) = f(x, 7) = x\sin(x + 7) \), which has a minimum of \( \pi \sin(\pi + 7) \) at \((\pi, 7)\) and a maximum of about 1.2 at about \((1.46, 7)\).
- On the right side, \( f(x, y) = f(\pi, y) = \pi \sin(\pi + y) \), which has minimum of \( -\pi \) at \((\pi, \pi/2)\) and a maximum of \( \pi \) at \((\pi, 3\pi/2)\).
- On the left side, \( f(x, y) = f(0, y) = 0 \).

Putting all this together, we see that the absolute maximum of \( \pi \) is achieved at \((\pi, 3\pi/2)\), while the minimum of \( -\pi \) is achieved at \((\pi, \pi/2)\).