1. Verify that divergence, curl, and gradient are linear operators.

*Solution.* For divergence, we want to show that for all vector fields \( \mathbf{F} \) and \( \mathbf{G} \) and scalars \( \alpha \) and \( \beta \), we have

\[
\nabla \cdot (\alpha \mathbf{F} + \beta \mathbf{G}) = \alpha \nabla \cdot \mathbf{F} + \beta \nabla \cdot \mathbf{G}.
\]

The left-hand side is

\[
\frac{\partial}{\partial x} (\alpha F_1 + \beta G_1) + \frac{\partial}{\partial y} (\alpha F_2 + \beta G_2) + \frac{\partial}{\partial z} (\alpha F_3 + \beta G_3) = \alpha \frac{\partial F_1}{\partial x} + \beta \frac{\partial G_1}{\partial x} + \alpha \frac{\partial F_2}{\partial y} + \beta \frac{\partial G_2}{\partial y} + \alpha \frac{\partial F_3}{\partial z} + \beta \frac{\partial G_3}{\partial z},
\]

which equals the right-hand side. Calculations for gradient and curl are similar.

2. Let \( \mathbf{F}(x, y, z) = (3x^2 + \frac{1}{2}y^2 + e^z, xy + z, f(x, y, z)) \). Find all \( f \) such that \( \mathbf{F} \) is curl-free.

*Solution.* The third component of \( \nabla \times \mathbf{F} \) is \( y - (1/2)(2y) = 0 \), as desired. For the first component to be zero, we must have \( f_y = 1 \), and for the second component to be zero we must have \( f_x = e^z \). Integrating these two equations tells us that \( f(x, y, z) = y + C_1(x, z) \) and \( f(x, y, z) = xe^z + C_2(y, z) \) for functions \( C_1 \) and \( C_2 \) which do not depend on \( y \) or \( x \) respectively. Putting these two together, we see that \( f(x, y, z) = xe^z + y + C(z) \) for any differentiable function \( C \).

3. Confirm that for a vector field \( \mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3 \), we have

\[
\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F},
\]

where \( \nabla^2 \mathbf{F} \) is defined to mean “take the Laplacian of each component of \( \mathbf{F} \).” Is it possible to derive this identity from \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \)?

*Solution.* [Omitted]

4. Let \( \mathbf{F} \) be a \( C^2 \) vector field on \( \mathbb{R}^3 \). Show that \( \nabla \times \mathbf{F} \) is incompressible.

*Solution.* We calculate

\[
\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0,
\]

since the mixed partials don’t depend on the order of differentiation, as \( \mathbf{F} \) is \( C^2 \).