1. Sketch the image of the path $\mathbf{x}(t) = (\cos t, \sin 2t)$.

**Solution.** As $t$ goes from 0 to $\pi/2$, the $x$-coordinate of $\mathbf{x}$ varies from 1 to 0, while the $y$-coordinate varies from 0 to 1 and back to 0. Superimposing these two “pen” movements (like an etch-a-sketch), we get a bump going from $(1,0)$ to $(0,0)$, as shown in the first quadrant below. Letting $t$ continue to increase produces the three more copies of this shape in the other three quadrants.

![Sketch of the path](image)

2. Find the arclength of the graph of $f(x) = \frac{2}{3}(x-1)^{3/2}$ between the points $(1,0)$ and $(4, 2\sqrt{3})$.

**Solution.** We calculate
$$
\int_1^4 \sqrt{1 + f'(x)^2} = \int_1^4 \sqrt{1 + (\sqrt{x-1})^2} \, dx = \frac{14}{3}.
$$

3. Consider the function $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\mathbf{F}(x, y, z) = (3x/y, 2x + e^z)$.

(a) Find $D\mathbf{F}$.

**Solution.** The total derivative is the matrix of partial derivatives:

$$
\begin{pmatrix}
\frac{3}{y} & -\frac{3x}{y^2} & 0 \\
2 & 0 & e^z
\end{pmatrix}
$$

(b) Show that there exists an open set $U \subset \mathbb{R}$ containing 1 and a function $\mathbf{f}: U \to \mathbb{R}^2$ such that for all $x \in U$, the equations $\mathbf{F}(x, y, z) = \mathbf{F}(1, -2, 0)$ have a unique solution $(y, z) = \mathbf{f}(x)$. Show that $\mathbf{f}$ is $C^1$.

**Solution.** The implicit function theorem ensures that we can solve (abstractly) for $(y, z)$ in terms of $x$ if the matrix of partial derivatives corresponding to the $y$ and $z$ columns has nonvanishing determinant. In this case, that means

$$
\det\begin{pmatrix}-3x/y^2 & 0 \\
0 & e^z
\end{pmatrix} = \det\begin{pmatrix}-3/4 & 0 \\
0 & 1
\end{pmatrix} = -3/4 \neq 0,
$$
so the implicit function theorem does apply and gives us the desired function $f$. The theorem also tells us that $f$ is $C^1$.

(c) Find $Df(1)$.

Solution. Since $F(x, f(x)) = c$ for some constant $c$, we can differentiate both sides to get $Df = -A^{-1}B$, where the matrix $A$ is the one obtained by taking the $y$ and $z$ columns of $DF$ and $B$ is the matrix obtained by considering the remaining columns. We get

$$Df(1) = \begin{pmatrix} -3/4 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

4. (from 3.2.15 in Colley) Determine the moving frame $(T, N, B)$, the curvature, the torsion, and the arclength parameter $s(t)$ for the curve

$$\mathbf{x} = \left( 5, \frac{1}{3}(t+1)^{3/2}, \frac{1}{3}(1-t)^{3/2} \right), \quad -1 < t < 1.$$

Solution. We have

$$T = \mathbf{x}'(t)/\|\mathbf{x}'(t)\| = \left( 1, \frac{1}{2}\sqrt{t+1}, -\frac{1}{2}\sqrt{1-t} \right)/\sqrt{3/2},$$

$$N = \frac{dT/dt}{\|dT/dt\|} = 2\sqrt{2} \left( 0, \frac{1}{2}\sqrt{1-t}, -\frac{1}{4}\sqrt{t+1} \right),$$

$$B = T \times N = (1, -\sqrt{t+1}, \sqrt{1-t})/\sqrt{3}.$$

The curvature and torsion are $\frac{\sqrt{2}}{6(1-t^2)}$ and $\frac{1}{3\sqrt{1-t^2}}$, respectively. The arclength parameter is $s(t) = \int_0^t \|\mathbf{x}'(\tau)\| \, d\tau = t/\sqrt{2}$. 