1. Let \( A = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix} \), and let \( U \) be the unit square \( \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\} \) in \( \mathbb{R}^2 \). Let \( U' \) be the image under \( A \) of \( U \). Find the area of \( U \).

Solution. Note that \( A = 2 \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} =: 2M \). The image of \( U \) under \( M \) is a parallelogram with unit base and height, and therefore it has unit area. The factor of 2 doubles the shape in both dimensions, giving a factor of 4 increase from the original area. So the area of \( U' \) is \( 1 \times 4 = 4 \).

2. Find the distance from the line \((4 + t, -1 - 2t, 3 - 7t)\) to the plane \( 3x - 2y + z = 3 \).

Solution. Since \((3, -2, 1) \cdot (1, -2, -7) = 0\), the line is parallel to the plane. Let \( P = (4, -1, 3) \) be a point on the line, and let \( Q \) be the point in the plane which is nearest to \( P \). Since \( \overrightarrow{QP} \) is parallel to the plane's normal vector \((3, -2, 1)\), we can write \( Q = P - \lambda (3, -2, 1) \) for some scalar \( \lambda \), substitute the resulting coordinates into the equation for the plane, and solve to find \( \lambda = 1 \). Therefore, the distance from the line to the plane is \( \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14} \).

3. Let \( A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & -2 \\ 5 & 0 \end{pmatrix} \). Find \( AB - BA \).

Solution. We calculate \( AB = \begin{pmatrix} -13 & -4 \\ 21 & -2 \end{pmatrix} \) and \( BA = \begin{pmatrix} 0 & -11 \\ 10 & -15 \end{pmatrix} \), so the difference \( AB - BA \) is \( \begin{pmatrix} -13 & 7 \\ 11 & 13 \end{pmatrix} \). Notice that this matrix measures the failure of \( A \) and \( B \) to commute.

4. Consider the function \( f(x, y, z) = (x^2 + y^2)/\sin(z) \). Describe the level surfaces for different values. What coordinate system is best suited for this?

Solution. Cylindrical coordinates are best suited, since \( x^2 + y^2 \) simplifies to \( r^2 \). The level surfaces \( \{(x, y, z) : f(x, y, z) = c\} \) are surfaces of revolution obtained by revolving a graph of \( r^2 = c \sin z \) (thought of as a curve in 2D) about the \( z \)-axis.

5. We say that a function \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is linear if \( f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) \). Characterize all linear functions from \( \mathbb{R} \) to \( \mathbb{R} \). Is \( f(x) = 7x - 4 \) linear, according to this definition?

Solution. Applying the definition of linearity with \( x = 1, y = 0 \), and \( \lambda \in \mathbb{R} \) arbitrary, we find that \( f(\lambda) = \lambda f(1) \). In other words, every linear function takes the form \( f(x) = mx \) for some constant \( m \). Conversely, every function of the form \( f(x) = mx \) is linear. Therefore, the linear functions are the ones whose graphs are lines passing through the origin.