1. Find the cosine of the angle between the vectors \((-4,2,6,5)\) and \((3,1,0,-7)\).

\textit{Solution.} The cosine of the angle between two vectors \(\mathbf{a}\) and \(\mathbf{b}\) is defined by \(\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| \cdot ||\mathbf{b}||} = \frac{-45}{9 \cdot \sqrt{59}} = -\frac{5}{\sqrt{59}}.

2. Is the cross-product associative? Is the dot product associative? Prove or give a counterexample for each.

\textit{Solution.} The cross-product is not associative. For example, \((\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0\) and \(\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = -\mathbf{j}\). The dot product is not associative because neither \((\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}\) nor \(\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})\) is defined.

3. Find an equation for the plane that contains the lines

\[
\begin{align*}
\begin{cases}
x(t) &= 5 + t \\
y(t) &= 1 - t \\
z(t) &= 4 - 3t.
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
x(t) &= 5 - 4t \\
y(t) &= 1 + t \\
z(t) &= 4 - 3t.
\end{cases}
\end{align*}
\]

\textit{Solution.} The plane contains the point \(P_0 = (5,1,4)\) and consists of all the points \(P = (x,y,z)\) for which \(\overrightarrow{P_0P} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = 0\), where \(\mathbf{v}_1 = (1,-1,-3)\) and \(\mathbf{v}_2 = (-4,1,-3)\). Since \(\mathbf{v}_1 \times \mathbf{v}_2 = (6,15,-3)\), the equation of the plane simplifies to \(6x + 15y - 3z = -33\).

4. Find the distance from the origin to the plane \(x + y + z = 1\).

\textit{Solution.} The shortest line segment from the origin \(O\) to the plane \(x + y + z = 1\) intersects the plane at some point \(P_0 = (x_0, y_0, z_0)\). Since \(\overrightarrow{PO}\) is perpendicular to the plane, the line containing \(\overrightarrow{PO}\) takes the form

\[
\begin{align*}
\begin{cases}
x(t) &= x_0 + t \\
y(t) &= y_0 + t \\
z(t) &= z_0 + t.
\end{cases}
\end{align*}
\]

Setting each of these coordinates equal to 0, we see that \(x_0 = y_0 = z_0\). Since the coordinates sum to 1, they are \((1/3,1/3,1/3)\). Thus the distance is \(\sqrt{(1/3)^2 + (1/3)^2 + (1/3)^2} = \sqrt{3}/3\).

We remark that it is also possible to solve this problem by computing the volume of the tetrahedron with vertices at \(O, (1,0,0), (0,1,0),\) and \((0,0,1)\) in two different ways.

5. Give a geometric description of what each of the following matrices does to a vector.

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad
B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad
C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad
D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
Solution. A does nothing, B increases the length by a factor of 3, C reflects across the y-axis, and D reflects across the line $y = x$.

6. What matrices have the following geometric descriptions?

(a) reflect across the x-axis

(b) reverse the direction of the vector and double its length

(c) halve the length of the vector (while preserving the direction)

(d) rotate a vector 90 degrees counterclockwise

(e) project a vector in $\mathbb{R}^3$ onto the x-y plane

Solution. (a) \[
\begin{pmatrix}-1 & 0 \\ 0 & 1 \end{pmatrix},
\]
(b) \[
\begin{pmatrix}-2 & 0 \\ 0 & -2 \end{pmatrix},
\]
(c) \[
\begin{pmatrix}1/2 & 0 \\ 0 & 1/2 \end{pmatrix},
\]
(d) \[
\begin{pmatrix}0 & -1 \\ 1 & 0 \end{pmatrix},
\]
(e) \[
\begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

7. The volume of a parallelepiped is the product of the area of its base and its height. Consider the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. You may suppose for simplicity that $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{a}$ form a right-handed triple of vectors, which means that a right-handed screw rotated an angle less than 180° from $\mathbf{b}$ to $\mathbf{c}$ advances in the direction of $\mathbf{a}$.

(a) Let us think of the parallelogram spanned by $\mathbf{b}$ and $\mathbf{c}$ as the base of the parallelepiped. What is the signed area of this parallelogram?

(b) What is the height of the parallelogram, in terms of $\mathbf{a}$ and the unit vector pointing in the direction of $\mathbf{b} \times \mathbf{c}$?

(c) Put together parts (a) and (b) to derive the triple scalar product formula for the volume of a parallelepiped.

Solution. The signed area of the parallelogram is $|\mathbf{b} \times \mathbf{c}|$. The height of the parallelogram is $\mathbf{a} \cdot \mathbf{u}$, where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{b} \times \mathbf{c}$. Therefore, the volume of the parallelepiped is

$$
\mathbf{a} \cdot (\mathbf{u} (\mathbf{b} \times \mathbf{c})) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).
$$

8. (Fun/Challenge problem) It is possible to prove the vector triple product formula

$$
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
$$

in a tedious way using coordinates. This problem outlines a more conceptual proof, taken from a note written by William C. Schulz.

(a) Use the parallelepiped interpretation of the triple scalar product to show that

$$
\mathbf{b} \cdot (\mathbf{c} \times \mathbf{n}) = \mathbf{c} \cdot (\mathbf{n} \times \mathbf{b}) = \mathbf{n} \cdot (\mathbf{b} \times \mathbf{c})
$$
(b) Use the right-hand rule to observe that if $c$ is perpendicular to $n$, then
\[ n \times (c \times n) = |n|^2 c. \]

(c) Show that it suffices to consider the case where $a$, $b$, and $c$ form a basis for $\mathbb{R}^3$.

(d) Write $a \times (b \times c)$ as a linear combination of $a$, $b$, and $c$, so that
\[ a \times (b \times c) = \kappa a + \lambda b + \mu c, \tag{1} \]

(e) The easiest coefficient to determine is $\kappa$. What is it?

(f) To find $\lambda$, dot both sides of (1) with $c \times n$, where $n := b \times c$.

(g) To find $\mu$, dot both sides of (1) with $b \times n$, where $n := b \times c$.