1. Consider a rotating cylindrical container of radius $R$ with a vertical axis, filled with liquid mercury up to height $h$. In this problem, we’ll use calculus of variations to find the shape of the surface of the liquid.

(a) Assume that the surface of the liquid is described by the equation $z = f(r)$ in cylindrical coordinates, where $f$ is some function. Sketch a graph of $f$ based on physical intuition. Indicate the locations of $R$ and $h$ on your axes.

(b) Calculate the potential energy $V$ of the liquid, assuming the density of the liquid is $\sigma$ and using $g$ to denote the gravitational constant. Express your answer as an integral involving $f$.

(c) Calculate the kinetic energy $T$ of the liquid, assuming the angular velocity of the cylinder is $\omega$. Express your answer as an integral involving $f$. 
(d) Calculate the volume of the liquid, as an integral in involving $f$.

(e) Hamilton’s principle says that the shape $f$ will minimize $T - V$. Also, a generalization of Lagrange multipliers along with the Euler-Lagrange equations implies the following. The extrema \( f(r) \) of \( \int_{r_1}^{r_2} G(r, f, f') \, dr \) subject to the constraint \( \int_{r_1}^{r_2} H(r, f, f') \, dr = C \) for some constant $C$ are solutions of the system

\[
\frac{d}{dr} \left( \frac{\partial(G - \lambda H)}{\partial f'} \right) = \frac{\partial(G - \lambda H)}{\partial f}
\]

\[
\int_{r_1}^{r_2} H(r, f, f') \, dr = C.
\]

Note that $f'$ is regarded as an independent variable. Put together the answers from the previous three parts to find $f$.

(f) Explain why your answer shows that the shape of the surface is independent of the density of the liquid.