1. Consider the function \( f(x) = \sqrt{1 - x^2} \) over the interval \([0, 1]\). Write down a definite integral whose value is equal to the arclength of the graph of \( f \).

**Solution.** We calculate an arclength of

\[
\int_0^1 \sqrt{1 + f'(x)^2} \, dx = \int_0^1 \sqrt{1 + \left( \frac{-2x}{2\sqrt{1-x^2}} \right)^2} \, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx.
\]

**Remark:** Since this arclength is \( \pi/2 \) by the definition of \( \pi \), this exercise proves that \( \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \pi/2 \). Moreover, it can be generalized to give a way of calculating the indefinite integral of \( 1/\sqrt{1-x^2} \).

2. Consider the function \( F : \mathbb{R}^4 \to \mathbb{R}^2 \) defined by \( F(w, x, y, z) = (2/w^2 - y, 3x + \cos z) \).

(a) Find \( DF \).

**Solution.** The total derivative is the matrix of partial derivatives:

\[
\begin{pmatrix}
-2/w^3 & 0 & -1 & 0 \\
0 & 3 & 0 & -\sin z
\end{pmatrix}
\]

(b) Show that there exists an open set \( U \subset \mathbb{R}^2 \) containing \((1, 2)\) and a function \( f : U \to \mathbb{R}^2 \) such that for all \( x \in U \), the equations \( F(w, x, y, z) = F(1, 2, 3, \pi/2) \) have a unique solution \((y, z) = f(w, x)\). Show that \( f \) is \( C^1 \).

**Solution.** The implicit function theorem ensures that we can solve (abstractly) for \((y, z)\) in terms of \( x \) and \( w \) if the matrix of partial derivatives corresponding to the \( y \) and \( z \) columns has nonvanishing determinant. In this case, that means

\[
\det \begin{pmatrix}
-1 & 0 \\
0 & -\sin z
\end{pmatrix} = \det \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix} = 1 \neq 0,
\]

so the implicit function theorem does apply. It ensures the existence of such an \( f \) and the fact that \( f \) is \( C^1 \).