If we want to integrate a function $f(x, y, z)$ over a region $R$ in $\mathbb{R}^3$, we choose some order of the variables $x$, $y$, and $z$, and we set up triple iterated integral, as follows (let’s work with the order $x-y-z$):

1. What’s the smallest and biggest that $x$ can be for any point in the region $R$? These are your outer limits of integration.

2. For any given value of $x$, what are the smallest and largest values that $y$ can be for any point in the region $R$ with that $x$ value. These are your bounds for the second integral, and they may depend on $x$.

3. For any given values of $x$ and $y$, what are the smallest and largest values that $z$ can be for any point in the region $R$ with those values of $x$ and $y$? These are your bounds for the innermost integral, and they may depend on $y$ and $x$.

If you choose a different order, then the process is the same but with the roles of $x$, $y$, and $z$ switched appropriately. Here’s an example.

**Example 1.** Set up a triple integral of a function $f(x, y, z)$ over a ball of radius 3 centered at $(0, 0, 0)$ in $\mathbb{R}^3$.

**Solution.** Let’s choose the order $y-z-x$. The smallest and largest values of $y$ for a point in the ball are $-3$ and $3$, so we start with

$$\int_{-3}^{3} dy.$$

Now, for any value of $y$, the intersection of the “$y = \text{constant}$” plane at that value of $y$ and the ball we’re integrating over is a disk of radius $\sqrt{9-y^2}$. The smallest and largest values of $z$ for a point in this disk are $-\sqrt{9-y^2}$ and $\sqrt{9-y^2}$, so those are our next limits of integration:

$$\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dz dy.$$

Finally, given both $y$ and $z$, we see that $(x, y, z)$ is inside the ball precisely when $x$ is between $-\sqrt{9-y^2-z^2}$ and $\sqrt{9-y^2-z^2}$. Thus, our integral becomes

$$\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-y^2-z^2}}^{\sqrt{9-y^2-z^2}} f(x, y, z) \, dx \, dz \, dy.$$

Once we know how to integrate over a region in \( \mathbb{R}^3 \), all the physics ideas we learned about laminas carry over to 3D regions. For example, the coordinates of the center of mass of a region \( R \) are equal to
\[
\left( \frac{\iiint_R x \, dV}{\iiint_R 1 \, dV}, \frac{\iiint_R y \, dV}{\iiint_R 1 \, dV}, \frac{\iiint_R z \, dV}{\iiint_R 1 \, dV} \right),
\]

note that \( \iiint_R 1 \, dV \) is equal to the volume of \( R \), just like \( \iint_L 1 \, dA \) is equal to the area of the lamina \( L \). This is because \( \iiint_R 1 \, dV \) says “chop the region \( R \) into a million tiny pieces, multiply the volume of each little piece by 1, and add up the results,” and if you divide up a region into pieces and add up their volumes, then you get the original volume back.

**Example 2.** Find the moment of inertia about the \( z \)-axis of the cylinder
\[
x^2 + y^2 \leq 1 \quad \text{and} \quad z^2 \leq 1,
\]
whose mass per unit volume at a point \((x, y, z)\) is given by the density function \( \rho(x, y, z) \).

**Solution.** Denote by \( R \) the cylinder described above. The moment of inertia is defined to be the integral over \( R \) of the mass \( \rho(x, y, z) \, dV \) (note that density times volume equals mass) times the squared distance to the axis of rotation, which in this case is \( x^2 + y^2 \), since we’re told that the \( z \)-axis is the axis of rotation. So we get
\[
\iint_R (x^2 + y^2) \rho(x, y, z) \, dV.
\]

We want to write this as an iterated integral, and let’s use the order \( z-x-y \). Clearly \( z \) ranges from \(-1\) to \(1\) over the region, so our outer integral is:
\[
\int_{-1}^{1} \, dz.
\]

For each value of \( z \), the “\( z = \text{constant} \)” slice intersects the region in a disk of radius 1, so \( x \) can be as small as \(-1\) and as large as 1. Thus we get
\[
\int_{-1}^{1} \int_{-1}^{1} \, dx \, dz.
\]

Finally, for each pair \((x, z)\), if we look at the line parallel to the \( y \)-axis which goes through \((x, 0, z)\), it intersects the region in the interval that starts at a \( y \)-value of \(-\sqrt{1-x^2}\) and ends at a \( y \)-value of \(\sqrt{1-x^2}\). Thus the total moment of inertia is
\[
\int_{-1}^{1} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2) \rho(x, y, z) \, dx \, dz,
\]
and that can’t be simplified any further since we don’t know the function \( \rho \). \(\square\)