Title: Koszul homology and classical invariant theory
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These are notes for a talk given at the commutative algebra seminar at University of Michigan on January 19, 2012.

1 Motivating problem.

1.1 Boij–Söderberg theory.

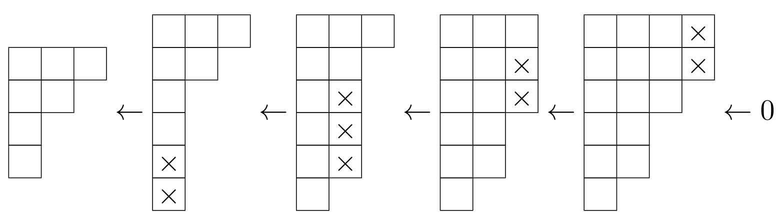
Our original motivation comes from trying to construct pure free resolutions over the homogeneous coordinate ring of a quadric hypersurface. First we review some definitions for polynomial rings.

Let $A = K[x_1, \dots, x_n]$ be a polynomial ring with the standard grading. Given a finitely generated graded A-module M, its tor modules are naturally graded, and we set $\beta_{i,j}(M) = \dim \operatorname{Tor}_i^A(M, K)_j$. A module M has a **pure free resolution** if $\operatorname{Tor}_i^A(M, K)$, when nonzero, is concentrated in a single degree $d_i(M)$ for all i (sometimes one imposes Cohen–Macaulayness of M). In this case, $d(M) = (d_0(M), d_1(M), \dots)$ is the **degree sequence** of M. Note that $d_0(M) < d_1(M) < \cdots$.

Theorem 1.1 (Eisenbud–Fløystad–Weyman [EFW], Eisenbud–Schreyer [ES])**.** *Given $d_0 < d_1 < \cdots < d_r$ (with $r \le n$), there is a module M with pure free resolution such that $d_i(M) = d_i$.*

One can make the same definition for quadric hypersurfaces $B = K[x_1, \dots, x_n]/q(x)$ where $q(x)$ is a homogeneous quadric. Note that minimal free resolutions over B are generally infinite in length, but become periodic of period 2 after n steps.

Our proposed approach is to transform the resolutions of Eisenbud–Fløystad–Weyman. The basic idea is the following. Write $A = \operatorname{Sym}(V)$, which naturally has an action of $G = \mathbf{GL}(V)$. The polynomial representations $\mathbf{S}_\lambda(V)$ of G are parametrized by partitions λ with at most n parts. We pictorially represent the free module $\mathbf{S}_\lambda(V) \otimes A$ as a Young diagram with λ_i boxes in the ith column. An example of an EFW complex for $n = 4$ is



The degree sequence is $(0, 2, 5, 7, 9)$. The general pattern: start with any partition λ, add arbitrarily many boxes to the first column for the next partition. At each additional step, add just enough boxes to the next column to get an overlap with the boxes you previously added. The differences in degrees is the number of boxes added each time.

1.2 Koike–Terada's universal character ring.

The character of $\mathbf{S}_\lambda(V)$ is the Schur polynomial $s_\lambda(x_1, \dots, x_n)$, and there is a well-defined way to take $n \to \infty$ and get Schur functions $s_\lambda(x_1, x_2, \dots)$. They have the property that $s_\lambda(x_1, \dots, x_n, 0, 0, \dots)$ are the Schur polynomials. This $n \to \infty$ case has an involution (as a ring) that doesn't exist for $n < \infty$: $\omega(s_\lambda) = s_{\lambda'}$ where λ' is the transpose partition.

Trying to generalize this setting to the character theory of the orthogonal and symplectic groups is more difficult. Koike–Terada [KT] found a way to do this, but there are some subtleties. We work with the orthogonal group so that we don't have to keep repeating both cases in what follows. Let V be an orthogonal space of dimension $2n$ or $2n+1$. First, polynomial representations $\mathbf{S}_{[\lambda]}(V)$ of $\mathbf{O}(V)$ are parametrized by partitions λ with at most n parts. The $n \to \infty$ character ring of the orthogonal group is defined as a subring of the ring of symmetric functions, and there is a restriction map π_n to the character ring of $\mathbf{O}(V)$ for all $n < \infty$. This has the property that $\pi_n(s_{[\lambda]}) = \mathrm{char}(\mathbf{S}_{[\lambda]}(V))$ if λ has at most n parts. Otherwise, it either gets sent to 0 (like for $\mathbf{GL}$), or a different irreducible character (with a sign). This universal character ring also has an involution as a ring: $\mathbf{KT}(s_{[\lambda]}) = s_{[\lambda']}$.

Something amazing happens at the $n \to \infty$ level. The character for $\mathrm{Sym}(V)$ is $\sum_{d \geq 0} s_d$ where d is a partition with 1 part equal to d. Formally, we can do this: apply ω to this character to get the character $\sum_{d \geq 0} s_{1^d}$ of the exterior algebra. Exterior powers remain irreducible for the orthogonal group, so upon restriction, the character becomes $\sum_{d \geq 0} s_{[1^d]}$. Now apply $\mathbf{KT}$ to get $\sum_{d \geq 0} s_{[d]}$, which is the character of the coordinate ring of the quadric hypersurface.

So if we carry out these formal manipulations to the EFW complexes (note that their definition for $n \to \infty$ makes sense by our description), we should get a pure free resolution over B. The main issue lies in making this precise, and moreso, in applying the restriction map π_n (which is not so easy to make sense of on the level of modules!)

1.3 Littlewood varieties.

We'll give a geometric / homological reinterpretation of what Koike–Terada's restriction maps mean.

Let E be a vector space and consider the subvariety of $\mathrm{Hom}(E, V)$ consisting of maps whose image is a totally isotropic subspace. We call this the **Littlewood variety**. This is set-theoretically defined by the quadratic equations $S^2E \otimes \mathbf{C} \subset S^2E \otimes S^2V^* \subset S^2(E \otimes V^*)$. The coordinate ring of this scheme is

$$\bigoplus_{\lambda} \mathbf{S}_\lambda E \otimes \mathbf{S}_{[\lambda]} V$$

where λ ranges over all partitions with at most $\min(\dim E, n)$ parts. If $\dim E \leq n$, then in fact, this scheme is reduced and a complete intersection. The Koszul complex on S^2E is then acyclic and looking at the parts that contain $\mathbf{S}_\lambda E$ for fixed λ will give a complex $\mathbf{F}^\lambda_\bullet$ of finite-dimensional vector spaces which only has homology in degree 0. Taking the character of this complex will express $\mathbf{S}_{[\lambda]}V$ as an alternating sum of $\mathbf{S}_\mu V$ for various μ (no square brackets). This gives Koike–Terada's expression of the universal character ring of $\mathbf{O}(V)$ as a subring of symmetric functions (at least for the partitions with at most n parts, and then consider larger n to get all of them).

Now if $\dim E > n$, the Koszul complex will have homology in general. But again considering the parts that contain $\mathbf{S}_\lambda E$ for fixed λ, we get a complex $\mathbf{F}^\lambda_\bullet$ of finite-dimensional vector spaces. The alternating sum will give the expression for $\pi_n(s_{[\lambda]})$ (after dividing by the character of $\mathbf{S}_\lambda E$). Koike–Terada's description of this as being either 0 or $\pm s_{[\mu]}$ for some μ suggests the following conjecture.

Conjecture 1.2. *$\mathbf{F}^\lambda_\bullet$ has homology in at most 1 degree.*

This is really a conjecture about the representation-theoretic structure of the Koszul homology of the scheme defined by S^2E above.

Similar remarks apply to the symplectic group, with S^2E replaced by $\bigwedge^2 E$.

2 Koszul homology.

The ideals we considered before are complicated for various reasons, so we started with easier examples from determinantal ideals to develop techniques for calculation and learn how much of the calculations can be done explicitly.

2.1 Strongly Cohen–Macaulay ideals.

An ideal I is **strongly Cohen–Macaulay** if the Koszul homology of a minimal generating set of I consists of Cohen–Macaulay modules. For determinantal ideals this only happens (except for trivial cases) for the $n \times n$ minors of a generic $n \times (n+1)$ matrix (Hilbert–Burch case) and the $2n \times 2n$ Pfaffians of a generic $(2n+1) \times (2n+1)$ skew-symmetric matrix (Buchsbaum–Eisenbud case).

Avramov–Herzog [AH] gave explicit minimal free resolutions for the Koszul homology of the Hilbert–Burch case. In fact, they have pure resolutions. They are also of geometric origin: they can be realized as pushforwards of vector bundles over a desingularization of the variety cut out by the minors, but we don't know an *a priori* reason for this other than knowing the minimal free resolutions first and finding a matching vector bundle.

Jerzy and I constructed minimal free resolutions for the Koszul homology of the Buchsbaum–Eisenbud case. These modules have **pure filtrations** (these are studied in [EES]): they have a filtration such that the quotient modules have pure resolutions. They don't seem to have a nice geometric origin.

2.2 Maximal minors.

For order n minors of an $n \times (n+k)$ matrix, we can completely describe the first Koszul homology group $\mathrm{H}_1 = \mathrm{H}_1(I)$. The basic idea is to study the exact sequence

$$\mathrm{H}_1 \to \mathrm{Tor}_1^A(A/I) \otimes A/I \to I/I^2 \to 0.$$

The kernel $\delta(I)$ fits into a short exact sequence

$$0 \to \delta(I) \to S^2(I) \to I^2 \to 0,$$

and ends up being the torsion submodule $\mathrm{H}_1^{\mathrm{tors}}$. Overall, H_1 has the following structure: $\mathrm{H}_1^{\mathrm{tors}}$ is supported on the order $n-1$ minors and the quotient $\mathrm{H}_1/\mathrm{H}_1^{\mathrm{tors}}$ has a filtration whose quotients are of geometric origin. If we iterate the process of taking torsion submodules, we get submodules supported on the order $n-3, n-5, n-7, \ldots$ minors and the quotients are all of geometric origin.

Similar behavior seems to happen for H_1 of the Littlewood ideals.

For the case $n = 2$, we can identify the torsion-free Koszul homology modules (they appear towards the end). They end up being modules of geometric origin. Note that the top Koszul homology is the canonical module of A/I.

In all cases above, the representation structure of the Koszul homology can be described explicitly.

2.3 Techniques.

For the modules of geometric origin, the desingularization is a vector bundle over a projective variety (product of Grassmannians). The "geometric technique" of Kempf–Lascoux–Weyman [Wey, Chapter 5] allows us to calculate the minimal free resolution of these modules.

To identify the torsion-free Koszul homology, we combine these resolutions with the Buchsbaum–Eisenbud acyclicity criterion.

To understand more of the intermediate Koszul homology, the theory of cotangent functors may be useful to produce more short exact sequences analogous to the one above for H_1.

References

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