Title: Free resolutions, degeneracy loci, and moduli spaces
Speaker: Steven Sam (joint work with Laurent Gruson and Jerzy Weyman)

These are notes for a talk given at the algebraic geometry seminar at University of Michigan on January 18, 2012.

1 An example.

Consider the space of $2 \times 2 \times 2$ complex arrays $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$. Note that we can realize these as the space of 2×2 matrices whose entries are linear forms in 2 variables. For "nondegenerate" arrays, the zero locus of the determinant is 2 points, and hence there isn't interesting moduli (if we consider equivalence via change of basis).

Now consider the space of $2 \times 2 \times 2$ integral arrays $U = \mathbf{Z}^2 \otimes \mathbf{Z}^2 \otimes \mathbf{Z}^2$. It carries a natural action of the group $G = \mathbf{SL}_2(\mathbf{Z}) \times \mathbf{SL}_2(\mathbf{Z}) \times \mathbf{SL}_2(\mathbf{Z})$ via change of coordinates. Then G acts on the coordinate ring $\mathrm{Sym}(U^*)$ of U and the subring of invariants is generated by a single polynomial f of degree 4. This is the simplest example of a hyperdeterminant and has the following explicit form. Fix a basis u_i, v_i for the ith factor of $\mathbf{Z}^2$. Then elements of U have expressions of the form

$$u_1 \otimes M_1 + v_1 \otimes N_1$$

where M_1 and N_1 are 2×2 matrices. Letting x, y be the dual coordinates on $\mathbf{Z}^2$, we can form the quadratic polynomial $Q_1(x, y) = \det(xM + yN)$. If we instead use the second or third factors of $\mathbf{Z}^2$, we get additional quadratic polynomials $Q_2(x, y)$ and $Q_3(x, y)$. The discriminants of these Q_i are all equal to f.

In [Bha], Bhargava defined a group law on primitive binary quadratic forms by declaring that

$$[Q_1] + [Q_2] + [Q_3] \equiv 0$$

whenever Q_1, Q_2, Q_3 come from the above construction.

Theorem 1.1 (Bhargava)**.** *There is a natural choice of origin which turns the set of primitive binary quadratic forms with given discriminant D into a group, and this group structure coincides with the Gauss composition law for primitive binary quadratic forms.*

To prove this, Bhargava constructed an explicit discriminant-preserving bijection between non-degenerate G-orbits in U and certain equivalence classes of pairs $(R, (I_1, I_2, I_3))$ where

- R is an "oriented" commutative ring with underlying additive group $\mathbf{Z}^2$
- I_1, I_2, I_3 are "oriented" fractional ideals with certain conditions

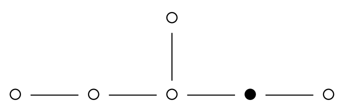
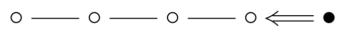
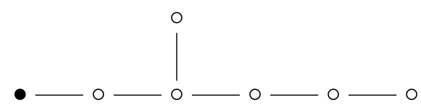
To generalize this situation, he considered the class of prehomogeneous vector spaces, i.e., those representations that have (G, U) such that $G(\mathbf{C})$ acts on $U(\mathbf{C})$ with an open orbit (in the above, we would use $G = \mathbf{GL}_2 \times \mathbf{GL}_2 \times \mathbf{GL}_2$). In such cases, the ring of semi-invariants $\mathrm{Sym}(U^*)^{(G,G)}$ is either trivial or a polynomial ring in one variable, and it is most interesting to consider $\mathbf{Z}$-orbits and find moduli interpretations for the orbits. Our work involves a different direction: considering those representations such that $\mathrm{Sym}(U^*)^{(G,G)}$ is a polynomial ring of dimension greater than 1 and finding moduli interpretations for the $G(\mathbf{C})$-orbits.

For the rest of the talk, we will work over $\mathbf{C}$, although it is possible most statements extend to fields of positive characteristic (we have not systematically figured out how much extends, so we omit this for simplicity).

2 Vinberg theory.

2.1 Representations of “finite type”.

Vinberg described a method in [Vin] for producing representations (G, U) which have finitely many orbits. I won’t describe the most general situation. Let Γ be a Dynkin diagram and pick a node $i \in \Gamma$. Let $\mathfrak{g}$ be the semisimple Lie algebra associated to $\Gamma \setminus i$. Let U be the representation of $\mathfrak{g}$ whose highest weight is the sum of fundamental weights corresponding to nodes adjacent to i. Some examples:

which correspond to

$$(\mathfrak{sl}_5(\mathbf{C}) \times \mathfrak{sl}_2(\mathbf{C}), \bigwedge^2 \mathbf{C}^5 \otimes \mathbf{C}^2), \qquad (\mathfrak{sl}_5(\mathbf{C}), S^2\mathbf{C}^5), \qquad (\mathfrak{so}_{12}(\mathbf{C}), \mathbf{spin}^+(12)).$$

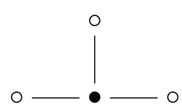
Integrate this action to a semisimple group and add a scaling $\mathbf{C}^*$ factor, and call the group G.

Theorem 2.1 (Vinberg)**.** *(G, U) has finitely many orbits.*

In the three examples, we get

- Pencils of 5×5 skew-symmetric matrices
- Symmetric 5×5 matrices
- Half-spinors of dimension 12

To get the example in the beginning of the talk, use the diagram



2.2 Representations of “affine type”.

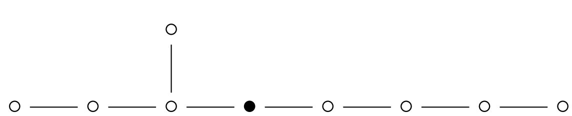
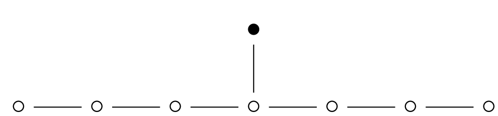
We can do the same as above where now Γ is an affine Dynkin diagram and get a pair (G, U) where G is a reductive group acting linearly on U.

Theorem 2.2 (Vinberg)**.** *$\mathrm{Sym}(U^*)^{(G,G)}$ is a polynomial ring, and the nullcone (zero locus of the positive degree (G, G) invariants) has finitely many G-orbits.*

We have been studying how to give modular interpretations for these orbit spaces. In all cases, they seem to be related to some kind of moduli space of curves (part of this investigation is also work in progress with Wei Ho and Jack Thorne).

Note that this generalizes the “finite type” case since when there are finitely many orbits, the nullcone is the whole space U. This also contains results of Dynkin and Kostant: if we choose i to be the “extending” vertex, then $U = \mathfrak{g}$ will be the adjoint representation, and this recovers the fact that the nilpotent cone is a complete intersection with finitely many orbits.

Some examples I will discuss in more detail:

These give the representations

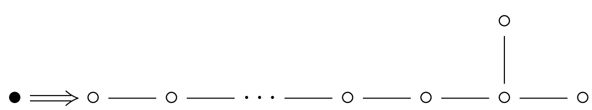
$$(\mathbf{SL}_5(\mathbf{C}) \times \mathbf{SL}_5(\mathbf{C}) \times \mathbf{C}^*, \bigwedge^2 \mathbf{C}^5 \otimes \mathbf{C}^5), \qquad (\mathbf{GL}_8(\mathbf{C}), \bigwedge^4 \mathbf{C}^8).$$

There are many examples, and there are many relations of the form: G' is a subgroup of G and U' is a G'-subrepresentation of U. Among those that are "maximal" there are some families and some sporadic examples. We study the sporadic examples and construct maps that take a "nondegenerate" vector $v \in U$ and produce some geometric object X that only depends on the G-orbit of v:

(G, U)	X
$(\mathbf{SL}_5(\mathbf{C}) \times \mathbf{SL}_5(\mathbf{C}) \times \mathbf{C}^*, \bigwedge^2 \mathbf{C}^5 \otimes \mathbf{C}^5)$	Elliptic curve torsor with degree 5 line bundle
$(\mathbf{GL}_9(\mathbf{C}), \bigwedge^3 \mathbf{C}^9)$	Abelian surface torsor with degree $(3,3)$ line bundle
$(\mathbf{GL}_8(\mathbf{C}), \bigwedge^4 \mathbf{C}^8)$	Abelian 3-fold torsor with degree $(2,2,2)$ line bundle
$(\mathbf{Spin}_{16}(\mathbf{C}), \mathbf{spin}^+(16))$	Abelian 4-fold torsor with degree $(2,2,2,2)$ line bundle

In the last three cases, the Abelian variety satisfies certain properties which we omit for simplicity of exposition. Furthermore, the correspondence $v \mapsto X$ appears to be a finite-to-1 map in each case. We also considered many "sub"examples (G', U') but we will omit their descriptions in the interest of time.

Example 2.3. Consider the Dynkin diagram of type $\mathrm{A}_{2n-1}^{(2)}$



This gives the pair $(\mathbf{SO}_{2n}(\mathbf{C}) \times \mathbf{C}^*, S_0^2\mathbf{C}^{2n})$. Here $S_0^2\mathbf{C}^{2n}$ is the quotient of $S^2\mathbf{C}^{2n}$ by the line spanned by the quadratic form q. Given a nondegenerate quadric $q' \in S_0^2\mathbf{C}^{2n}$, we can form the pencil $xq + yq'$. The determinant gives us $2n$ points in $\mathbf{P}^1$ and hence a hyperelliptic curve C of genus $n-1$, and this process is reversible. This situation was considered by Weil.

Desale and Ramanan [DR] showed the following. Consider the intersection of the quadrics defined by q and q' in $\mathbf{P}^{2n-1}$. Then the variety of $\mathbf{P}^{n-2}$'s in $q \cap q'$ is a torsor for the Jacobian of C, and the variety of $\mathbf{P}^{n-3}$'s in $q \cap q'$ is isomorphic to the moduli space of rank 2 stable vector bundles with fixed odd determinant ξ.

These constructions can be interpreted as "degeneracy loci" as follows. Consider the orthogonal Grassmannian $\mathbf{OGr}(n-1, 2n)$, which is the subvariety of $\mathbf{Gr}(n-1, 2n)$ whose points are totally isotropic subspaces for q. The trivial bundle $\mathbf{Gr}(n-1, 2n) \times \mathbf{C}^{2n}$ has a tautological rank $n-1$ subbundle $\mathcal{R} = \{(x, W) \mid x \in W\}$. Then we have $S_0^2\mathbf{C}^{2n} = \mathrm{H}^0(\mathbf{OGr}(n-1, 2n); S^2\mathcal{R}^*)$ and q' gives a generic section whose zero locus is the variety of $\mathbf{P}^{n-2}$'s in $q \cap q'$. Similar comments apply to the variety of $\mathbf{P}^{n-3}$'s using $\mathbf{OGr}(n-2, 2n)$.

Modular interpretations for the degeneracy loci for the other Grassmannians $\mathbf{OGr}(k, 2n)$ are given by Ramanan [Ram, §6, Theorem 3]. □

The general idea is as follows: using the Borel–Weil construction find a homogeneous space G/P and a homogeneous vector bundle $\mathcal{U}$ such that $U = \mathrm{H}^0(G/P; \mathcal{U})$ as G-modules. In all cases, over any point $x \in G/P$, the stabilizer of x will act on the fiber $\mathcal{U}|_x$ with finitely many orbits (in fact, we can take a reductive subgroup of the stabilizer). These fibers are essentially the same for any x, so we can glue together these orbits. If we pick a section $v \in U$, global orbits will give us "degeneracy loci", which are related to X in some way.

In all cases, we can construct perfect resolutions supported on these orbit closures (a resolution of length ℓ is **perfect** if it resolves a module whose annihilator contains a regular sequence of length ℓ; in our case, this is the same as saying that the codimension of the support is ℓ). Usually they are free resolutions of the coordinate ring. These translate to locally free resolutions of some coherent sheaf supported on the degeneracy locus (usually the structure sheaf) via the Eagon–Northcott generic perfection theorem. By calculating cohomological invariants of the degeneracy locus, we can usually gain a lot of information about it.

The choice of P is certainly not unique, so comparing different choices gives us some interesting geometric relations to play around with.

3 $\bigwedge^2 \mathbf{C}^5 \otimes \mathbf{C}^5$

Let A and B be 5-dimensional vector spaces. Take $G = \mathbf{SL}_5(A) \times \mathbf{SL}_5(B) \times \mathbf{C}^*$ and $U = A \otimes \bigwedge^2 B$. The simplest way to study this example is as follows.

Take $G/P = \mathbf{P}(A^*)$ (lines in A^*) and $\mathcal{U} = \mathcal{O}(1) \otimes \bigwedge^2 \underline{B}$ (here $\underline{B} = \mathbf{P}(A^*) \times B$ is a trivial bundle). Then $U = \mathrm{H}^0(\mathbf{P}(A^*); \mathcal{O}(1) \otimes \bigwedge^2 \underline{B})$. The fibers all look like the space of 5×5 skew-symmetric matrices, so the degeneracy loci are the Pfaffian loci of $\mathcal{U}$. For a generic section $v \in U$, the only nonempty locus C is the one cut out by the 4×4 Pfaffians, and this will be smooth of codimension 3. Now consider the Buchsbaum–Eisenbud complex

$$0 \to \mathcal{O}(-5) \to \underline{B}^* \otimes \mathcal{O}(-3) \to \bigwedge^4 \underline{B}^* \otimes \mathcal{O}(-2) \to \mathcal{O}_{\mathbf{P}(A^*)} \to \mathcal{O}_C \to 0.$$

which is a locally free resolution of $\mathcal{O}_C$. From this resolution, we can calculate $\mathrm{h}^0(\mathcal{O}_C) = \mathrm{h}^1(\mathcal{O}_C) = 1$, which say that C is connected and has genus 1. Choosing a basepoint on C turns it into an elliptic curve, and the restriction of $\mathcal{O}(1)$ to C gives the degree 5 line bundle. Basic properties of elliptic curves and the Buchsbaum–Eisenbud classification of codimension 3 Gorenstein ideals [BE, Theorem 2.1] shows that every elliptic curve C with a degree 5 line bundle arises from the above construction.

Although this example was rather simple, it is interesting to consider other choices of G/P. One possibility is to replace it with a different Grassmannian $\mathbf{Gr}(k, A^*)$ (where the above is $k = 1$). The trivial bundle $A^* \times \mathbf{Gr}(k, A^*)$ contains a tautological rank k subbundle $\mathcal{R} = \{(a, A') \mid a \in A'\}$, and one can show that $\mathrm{H}^0(\mathbf{Gr}(k, A^*); \mathcal{R}^*) = A$ as $\mathbf{GL}(A)$-modules (this is a special case of the Borel–Weil theorem, which guides the more difficult examples), so we can take $\mathcal{U} = \mathcal{R}^* \otimes \bigwedge^2 \underline{B}$.

First consider $k = 2$. The fibers of $\mathcal{U}$ are $\mathbf{C}^2 \otimes \bigwedge^2 \mathbf{C}^5$. The smallest orbit is the affine cone over $\mathbf{P}^1 \times \mathbf{Gr}(2, 5)$. Its tangential variety has codimension 5 and its secant variety has codimension 5. If we glue together these orbit closures and consider the corresponding degeneracy loci in $\mathbf{Gr}(2, A^*)$ using the same section as before, we get subvarieties T and S of codimensions 5 and 4, respectively.

Proposition 3.1. *T is the locus of 2-planes that are tangent to some point of C, and S is the locus of 2-planes W such that $\deg(\mathbf{P}(W) \cap C) \geq 2$. Furthermore, the map $\psi \colon C \to T$ sending $x \in C$ to its tangent line $\mathrm{T}_x(C)$ is an isomorphism.*

Now consider $k = 3$. The fibers of $\mathcal{U}$ are $\mathbf{C}^3 \otimes \bigwedge^2 \mathbf{C}^5$. The $\mathbf{SL}_3 \times \mathbf{SL}_5$-invariants of this space are generated by a single invariant of degree 15. The corresponding degeneracy locus in $\mathbf{Gr}(3, A^*)$ is a hypersurface of degree 5.

Proposition 3.2. *This hypersurface is the cut out by the Chow form of C, i.e., it consists of all 3-planes W such that $\mathbf{P}(W) \cap C \neq \varnothing$.*

Finally, consider $k = 4$. The fibers of $\mathcal{U}$ are $\mathbf{C}^4 \otimes \bigwedge^2 \mathbf{C}^5$. The $\mathbf{SL}_4 \times \mathbf{SL}_5$-invariants of this space are generated by a single invariant of degree 40. The corresponding degeneracy locus in $\mathbf{Gr}(4, A^*) = \mathbf{P}(A)$ is a hypersurface of degree 10.

Proposition 3.3. *This hypersurface is the projective dual of C, i.e., it consists of all hyperplanes in $\mathbf{P}(A^*)$ which are tangent to C.*

4 $\bigwedge^4 \mathbf{C}^8$.

We will give two constructions which will lead to the same geometric object.

Let V be an 8-dimensional vector space and take $U = \bigwedge^4 V$.

Take $G/P = \mathbf{P}(V^*)$ and let $\mathcal{Q} = \underline{V}^*/\mathcal{O}(-1)$ be the rank 7 quotient bundle. Then we have $V = \mathrm{H}^0(\mathbf{P}(V^*); \bigwedge^3 \mathcal{Q}^* \otimes \mathcal{O}(1))$, so we set $\mathcal{U} = \bigwedge^3 \mathcal{Q}^* \otimes \mathcal{O}(1)$.

The fibers of $\mathcal{U}$ are $\bigwedge^3 \mathbf{C}^7$. This has a degree 7 $\mathbf{SL}_7$-invariant. The singular locus X' of the corresponding hypersurface has codimension 4 and is the closure of a single $\mathbf{GL}_7$-orbit. In fact, it is Cohen–Macaulay and we can calculate its minimal free resolution using the geometric technique of Kempf–Lascoux–Weyman [Wey, Chapter 5]. The singular locus Z' of X' has codimension 7 and similar remarks apply to it. In fact, the ideal of Z' in $\mathcal{O}_{X'}$ is the symmetric square of the canonical module of X'. Using this, we can construct an algebra structure on $\mathcal{O}_{X'} \oplus \omega_{X'}$, define $\widetilde{X}'$ to be its spectrum, and get a degree 2 map $\pi \colon \widetilde{X}' \to X'$.

After picking a generic section $v \in U$, this gives rise to a map $\pi \colon \widetilde{X} \to X$ of 3-dimensional varieties. The locally free resolution of $\mathcal{O}_X$ is given by

$$0 \to \mathcal{Q} \otimes \mathcal{O}(-7) \to \bigwedge^2 \mathcal{Q} \otimes \mathcal{O}(-6) \to \mathrm{ad}(\mathcal{Q}) \otimes \mathcal{O}(-4) \to \bigwedge^4 \mathcal{Q} \otimes \mathcal{O}(-4) \to \mathcal{O}$$

where $\mathrm{ad}(\mathcal{Q}) = \ker(\mathcal{Q} \otimes \mathcal{Q}^* \to \mathcal{O})$. Using this minimal free resolution, one can calculate $\mathrm{h}^i(\mathcal{O}_{\widetilde{X}}) = \binom{3}{i}$ and $\omega_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}$. Now we can use the following result, which is a consequence of a theorem of Kawamata [Kaw, Corollary 2]:

Theorem 4.1 (Kawamata)**.** *Let Y be a smooth, connected, projective complex variety of dimension g such that $\omega_Y = \mathcal{O}_Y$ and $\mathrm{h}^1(\mathcal{O}_Y) = g$. Then Y is an Abelian variety (after picking a basepoint).*

Hence $\widetilde{X}$ is a torsor over an Abelian 3-fold, and one can also show that X is its Kummer quotient. The pullback of the line bundle $\mathcal{O}(1)$ on $\mathbf{P}(V^*)$ gives the degree $(2, 2, 2)$ line bundle on $\widetilde{X}$.

Remark 4.2. The degree 7 $\mathbf{SL}_7$-invariant also gives a degeneracy locus which is a quartic hypersurface in $\mathbf{P}(V^*)$. This is known as the Coble quartic, and it is the unique quartic which is singular along X (see [Bea] for more details). □

There is a way to construct $\widetilde{X}$ as a subvariety. We need to work with the partial flag variety $\mathbf{Fl}(1, 7, V^*)$, which is the incidence locus in $\mathbf{P}(V^*) \times \mathbf{Gr}(7, V^*)$ consisting of pairs of lines and hypersurfaces (ℓ, H) such that $\ell \subset H$. This naturally has two subbundles $\mathcal{R}_1$ and $\mathcal{R}_7$ obtained by pulling back the tautological subbundles from $\mathbf{P}(V^*)$ and $\mathbf{Gr}(7, V^*)$. Using Borel–Weil, one has

$$\bigwedge^4 V = \mathrm{H}^0(\mathbf{Fl}(1, 7, V^*); \mathcal{R}_1^* \otimes \bigwedge^3 (\mathcal{R}_7/\mathcal{R}_1)^*).$$

The fibers of this bundle are $\bigwedge^3 \mathbf{C}^6$, and the orbit of interest is the affine cone over $\mathbf{Gr}(3, 6)$, or the locus of pure tensors.

Proposition 4.3. *The corresponding degeneracy locus is $\widetilde{X}$ and the restriction of the forgetful map $\mathbf{Fl}(1, 7, V^*) \to \mathbf{P}(V^*)$ is the map π.*

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