Title: Free resolutions, degeneracy loci, and moduli spaces
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These are notes for a talk given at Manjul Bhargava's seminar on "Arithmetic invariant theory" at Princeton University on November 21, 2011.

The purpose of this talk is to explain how one can transform knowledge about certain free resolutions and vector bundles into descriptions for certain moduli spaces of Abelian varieties. The main tools are the Eagon–Northcott generic perfection theorem, the Borel–Weil–Bott theorem, and Kawamata's birational characterization of Abelian varieties. We end the talk with a few examples.

1 Free resolutions.

A lot of the foundational results in this section can be found in [BV, Appendix].

Definition 1.1. Let R be a commutative ring and M be a finitely generated R-module. A complex of R-modules

$$\mathbf{F}_\bullet : \cdots \to \mathbf{F}_i \xrightarrow{d_i} \mathbf{F}_{i-1} \to \cdots \to \mathbf{F}_0$$

is a **projective resolution** of M if

- Each $\mathbf{F}_i$ is a finitely generated projective R-module,
- $\mathrm{H}_i(\mathbf{F}_\bullet) = 0$ for $i > 0$ and $\mathrm{H}_0(\mathbf{F}_\bullet) = M$,

The **projective dimension** of M (denoted $\operatorname{pdim} M$) is the minimum length of any projective resolution of M. **Free resolutions** are defined by using free modules. □

Example 1.2. Let K be a commutative ring and let E be a free K-module of rank n and write $A = \operatorname{Sym}(E)$. We think of A as a graded ring via degree of polynomials. Then A_i is the degree i part of A, and $A(d)$ denotes A with a grading shift: $A(d)_i = A_{d+i}$. We define a complex $\mathbf{F}_\bullet$ by setting $\mathbf{F}_i = \bigwedge^i E \otimes A(-i)$. The differential is defined by

$$\bigwedge^i E \otimes A(-i) \to \bigwedge^{i-1} E \otimes A(-i+1)$$

$$e_1 \wedge \cdots \wedge e_i \otimes f \mapsto \sum_{j=1}^{i} (-1)^j e_1 \wedge \cdots \hat{e_j} \cdots \wedge e_i \otimes e_j f.$$

This is the **Koszul complex**. It is a resolution of $K = A/\mathfrak{m}$, and is functorial with respect to E. □

Definition 1.3. Let R be a Noetherian ring and let M be a finitely generated R-module. A sequence $(r_1, \dots, r_n)$ of elements in R is a **regular sequence** on M if

- r_1 is not a zerodivisor or unit on M, and
- r_i is not a zerodivisor or unit on $M/(r_1, \dots, r_{i-1})M$ for all $i > 1$.

For an ideal $I \subset R$, The **depth** of M (with respect to I) is the length of the longest regular sequence for M which is contained in I. It is denoted by $\operatorname{depth}_I M$. If R is local with maximal ideal $\mathfrak{m}$, we denote $\operatorname{depth} M = \operatorname{depth}_{\mathfrak{m}} M$. For an ideal $I \subset R$, the **grade** of I is the length of the longest regular sequence in I for R. M is **perfect of grade** g if $g = \operatorname{pdim} M = \operatorname{grade} \operatorname{Ann} M$. (In general, one has $\operatorname{pdim} M \geq \operatorname{grade} \operatorname{Ann} M$.)

Over a local Noetherian ring R, a finitely generated module M is **Cohen–Macaulay** if it is 0 or $\operatorname{depth} M = \dim M := \dim(R/\operatorname{Ann}(M))$. For a general Noetherian ring R, M is Cohen–Macaulay if the localization $M_\mathfrak{p}$ is Cohen–Macaulay over $(R_\mathfrak{p}, \mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R. A Noetherian ring is Cohen–Macaulay if it is a Cohen–Macaulay module over itself. □

Theorem 1.4. *Let R be a Noetherian Cohen–Macaulay ring.*
1. *For every ideal $I \subset R$, we have* $\operatorname{grade} I = \operatorname{codim} I = \dim R - \dim(R/I)$.
2. *The polynomial ring $R[x]$ is Cohen–Macaulay.*
3. *If an R-module M is perfect, then it is Cohen–Macaulay.*

(The distinction between perfect and Cohen–Macaulay for a module over a Cohen–Macaulay ring is the property having finite projective dimension.)

Theorem 1.5 (Auslander–Buchsbaum)**.** *Suppose R is a local Noetherian ring and that M is a finitely generated R-module with* $\operatorname{pdim} M < \infty$. *Then*

$$\operatorname{depth} M + \operatorname{pdim} M = \operatorname{depth} R.$$

Furthermore, the above is true for $R = K[x_1, \dots, x_n]$ with K a field and graded modules M.

Theorem 1.6. *Let R be a Noetherian local ring and M be a perfect R-module of grade g with minimal free resolution $\mathbf{F}_\bullet$. Then* $\operatorname{Hom}(\mathbf{F}_\bullet, R)$ *is a minimal free resolution of the perfect module* $M^\vee = \operatorname{Ext}^g_R(M, R)$, *and* $(M^\vee)^\vee \cong M$.

Definition 1.7. If $M = R/I$, for an ideal I, is perfect, then we write $\omega_{R/I} = M^\vee$ and call it the **canonical module** of R/I. If R/I is perfect and $\omega_{R/I} \cong R/I$ (ignoring grading if it is present), then we say that I is a **Gorenstein ideal**. This is equivalent to the last term in the minimal free resolution of R/I having rank 1. □

Theorem 1.8 (Eagon–Northcott generic perfection)**.** *Let R be a Noetherian ring and M a perfect R-module of grade g, and let $\mathbf{F}_\bullet$ be an R-linear free resolution of M of length g. Let S be a Noetherian R-algebra. If $M \otimes_R S \neq 0$ and* $\operatorname{grade}(M \otimes_R S) \geq g$, *then $M \otimes_R S$ is perfect of grade g and $\mathbf{F}_\bullet \otimes_R S$ is an S-linear free resolution of $M \otimes_R S$. If $M \otimes_R S = 0$, then $\mathbf{F}_\bullet \otimes_R S$ is exact.*

See [BV, Theorem 3.5].

Remark 1.9. In particular, if K is Cohen–Macaulay and $R = A$, then we can replace grade in the above theorem with codimension. It is often much easier to calculate codimension. We will use it as follows: Let K be a field or $\mathbf{Z}$. We construct graded minimal free resolutions of Cohen–Macaulay modules M over $A = K[x_1, \dots, x_n]$. Then we specialize the variables x_i to elements of a Cohen–Macaulay K-algebra S in such a way that the codimension of M is preserved. Then the resulting specialized complex is still a resolution. □

Example 1.10. Let K be any commutative ring, and let E be a free K-module of rank $2n + 1$. Write $\det E = \bigwedge^{2n+1} E$. Set $A = \operatorname{Sym}(\bigwedge^2 E)$, which we can interpret as the coordinate ring of the space of all skew-symmetric matrices of size $2n+1$ with entries in K if we fix a basis $e_1, \dots, e_{2n+1}$ of E. Let Φ be the generic skew-symmetric matrix of size $2n+1$ whose (i, j) entry is $x_{ij} = e_i \wedge e_j \in A_1$. We construct a complex

$$\mathbf{F}_\bullet : 0 \to (\det E)^{\otimes 2} \otimes A(-2n-1) \to (\det E) \otimes E \otimes A(-n-1) \to \bigwedge^{2n} E \otimes A(-n) \to A$$

For $j = 1, \dots, 2n + 1$, let $e'_j = e_1 \wedge \cdots \hat{e}_j \cdots \wedge e_{2n+1}$. We also define $\operatorname{Pf}(\hat{\jmath})$ to be the Pfaffian of the submatrix of Φ obtained by deleting row and column j. Then we have

$$\begin{aligned}
\bigwedge^{2n} E \otimes A(-n) &\xrightarrow{d_1} A \\
e'_j \otimes f &\mapsto \mathrm{Pf}(\hat{\jmath})f \\
(\det E) \otimes E \otimes A(-n-1) &\xrightarrow{d_2} \bigwedge^{2n} E \otimes A(-n) \\
(e_1 \wedge \cdots \wedge e_{2n+1}) \otimes e_j \otimes f &\mapsto \sum_{i=1}^{2n+1} (-1)^i e'_i \otimes x_{ij} f \\
(\det E)^{\otimes 2} \otimes A(-2n-1) &\xrightarrow{d_3} (\det E) \otimes E \otimes A(-n-1) \\
(e_1 \wedge \cdots \wedge e_{2n+1})^2 \otimes f &\mapsto (e_1 \wedge \cdots \wedge e_{2n+1}) \otimes \sum_{j=1}^{2n+1} (-1)^j \mathrm{Pf}(\hat{\jmath}) e_j f
\end{aligned}$$

This is the **Buchsbaum–Eisenbud complex**. It is a resolution of A/I where I is the ideal generated by the $2n \times 2n$ Pfaffians of Φ, and it is functorial with respect to E. Furthermore, A/I is a free K-module, and I is a Gorenstein ideal of codimension 3. We can identify d_2 with the map Φ. Buchsbaum and Eisenbud showed that given a codimension 3 Gorenstein ideal I, there is an n such that its free resolution is a specialization of the above complex. See [BE] for more details. □

2 Vector bundles.

2.1 Schur functors.

For the material in this section, see [Wey, Chapter 2]. What we call $\mathbf{S}_\lambda$ is denoted by $\mathbf{L}_{\lambda'}$ there.

Definition 2.1. A partition λ is a decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We represent this as a Young diagram by drawing λ_i boxes left-justified in the ith row, starting from top to bottom. The dual partition λ' is obtained by letting λ'_i be the number of boxes in the ith column of λ. Given a box $b = (i,j) \in \lambda$, its **content** is $c(b) = j - i$ and its **hook length** is $h(b) = \lambda_i - i + \lambda'_j - j + 1$. □

Example 2.2. Let $\lambda = (4,3,1)$. Then $\lambda' = (3,2,2,1)$. The contents and hook lengths are given as follows:

$$c: \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \\ \cline{1-3} -2 & & & \\ \cline{1-1}\end{array} \qquad h: \begin{array}{|c|c|c|c|}\hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 & \\ \cline{1-3} 1 & & & \\ \cline{1-1}\end{array}$$

□

Definition 2.3. Let R be a commutative ring and E a free R-module. Let λ be a partition with n parts and write $m = \lambda_1$. We use $S^n E$ to denote the nth symmetric power of E. The **Schur functor** $\mathbf{S}_\lambda(E)$ is the image of the map

$$\bigwedge^{\lambda'_1} E \otimes \cdots \otimes \bigwedge^{\lambda'_m} E \xrightarrow{\Delta} E^{\otimes \lambda'_1} \otimes \cdots \otimes E^{\otimes \lambda'_m} = E^{\otimes \lambda_1} \otimes \cdots \otimes E^{\otimes \lambda_n} \xrightarrow{\mu} S^{\lambda_1} E \otimes \cdots \otimes S^{\lambda_n} E,$$

where the maps are defined as follows. First, Δ is the product of the comultiplication maps $\bigwedge^i E \to E^{\otimes i}$ given by $e_1 \wedge \cdots \wedge e_i \mapsto \sum_{w \in S_i} \mathrm{sgn}(w) e_{w(1)} \otimes \cdots \otimes e_{w(i)}$. The equals sign is interpreted as follows: pure tensors in $E^{\otimes \lambda'_1} \otimes \cdots \otimes E^{\otimes \lambda'_m}$ can be interpreted as filling the Young diagram of λ

with vectors along the columns, which can be thought of as pure tensors in $E^{\otimes \lambda_1} \otimes \cdots \otimes E^{\otimes \lambda_n}$ by reading via rows. Finally, μ is the multiplication map $E^{\otimes i} \to S^i E$ given by $e_1 \otimes \cdots \otimes e_i \mapsto e_1 \cdots e_i$.

In particular, note that $\mathbf{S}_\lambda E = 0$ if the number of parts of λ exceeds $\operatorname{rank} E$. □

Example 2.4. Take $\lambda = (3, 2)$. Then the map is given by

$$\begin{aligned} (e_1 \wedge e_2) \otimes (e_3 \wedge e_4) \otimes e_5 \quad &\mapsto \begin{array}{|c|c|c|}\hline e_1 & e_3 & e_5 \\ \hline e_2 & e_4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline e_2 & e_3 & e_5 \\ \hline e_1 & e_4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline e_1 & e_4 & e_5 \\ \hline e_2 & e_3 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline e_2 & e_4 & e_5 \\ \hline e_1 & e_3 \\ \cline{1-2}\end{array} \\ &\mapsto (e_1 e_3 e_5 \otimes e_2 e_4) - (e_2 e_3 e_5 \otimes e_1 e_4) - (e_1 e_4 e_5 \otimes e_2 e_3) + (e_2 e_4 e_5 \otimes e_1 e_3) \end{aligned}$$

□

Theorem 2.5. *The Schur functor $\mathbf{S}_\lambda E$ is a free R-module. If $\operatorname{rank} E = n$, then*

$$\operatorname{rank} \mathbf{S}_\lambda E = \prod_{b \in \lambda} \frac{n + c(b)}{h(b)}.$$

The construction of $\mathbf{S}_\lambda E$ is functorial with respect to E. This has two consequences: $\mathbf{S}_\lambda E$ is naturally a representation of $\mathbf{GL}(E)$, and we can also construct $\mathbf{S}_\lambda \mathcal{E}$ when $\mathcal{E}$ is a vector bundle.

2.2 Borel–Weil theorem.

For the material in this section, see [Wey, Chapter 4.1].

(For simplicity, we work over a field, though everything in this section works over an arbitrary base scheme.)

Let V be an n-dimensional vector space and let $\mathbf{Gr}(k, V)$ denote the Grassmannian of k-dimensional *subspaces* of V. There is a tautological exact sequence of vector bundles

$$0 \to \mathcal{R} \to V \times \mathbf{Gr}(k, V) \to \mathcal{Q} \to 0$$

where $\mathcal{R} = \{(v, W) \mid v \in W\}$ has rank k and $\mathcal{Q}$ has rank $n - k$.

Remark 2.6. When $k = 1$, then $\mathbf{Gr}(k, V) \cong \mathbf{P}^{n-1}$ and $\mathcal{R} = \mathcal{O}(-1)$. □

These bundles carry an action of $\mathbf{GL}(V)$ compatible with the action on $\mathbf{Gr}(k, V)$.

Theorem 2.7 (Borel–Weil)**.** *Let λ be a partition with at most n parts. There is a $\mathbf{GL}(V)$-equivariant isomorphism*

$$\mathrm{H}^0(\mathbf{Gr}(k, V); \mathbf{S}_{(\lambda_1, \ldots, \lambda_k)}(\mathcal{R}^*) \otimes \mathbf{S}_{(\lambda_{k+1}, \ldots, \lambda_n)}(\mathcal{Q}^*)) = \mathbf{S}_\lambda(V^*).$$

2.3 Degeneracy loci.

First, we point out that the notions of Cohen–Macaulay generalize to varieties and coherent sheaves. Also, the generic perfection theorem still makes sense.

We'll be interested in the following situation. Let X be a variety (usually a Grassmannian) with a vector bundle $\mathcal{E}$ over X. We'll construct certain subvarieties Y in the total space of $\mathcal{E}$ for which we can construct (locally) free resolutions that fit the hypotheses of the generic perfection theorem (i.e., length of the resolution is the grade of the ideal sheaf of Y).

Given a section $v \in \mathrm{H}^0(X; \mathcal{E})$, we are interested in the subvarieties $v(X) \cap Y$, and in particular, when the grade of the ideal sheaf does not change. Then this gives a locally free resolution of $v(X) \cap Y \subset X$, and this will allow us to read off properties of this variety.

In particular, we can try to use this resolution to calculate the canonical sheaf of $v(X) \cap Y$ and the cohomology of its structure sheaf. Here is why we want to be able to do this:

Theorem 2.8. *Let X be a g-dimensional geometrically connected projective nonsingular variety over a field of characteristic 0. If $\omega_X \cong \mathcal{O}_X$ and $\dim \mathrm{H}^1(X;\mathcal{O}_X) = g$, then X is a torsor over an Abelian variety (namely, its Albanese variety).*

This can be deduced from [Kaw, Corollary 2].

Remark 2.9. This theorem already fails for $g = 2$. In particular, it is valid if the characteristic is different from 2 or 3, but in these small characteristics, there are new exotic examples, known as quasi-hyperelliptic surfaces which come from the Bombieri–Mumford classification of surfaces (the quasi-hyperelliptic surfaces have the property that their Picard varieties are non-reduced), see [BM, p.25, Table]. □

3 Examples.

For now, we work over the complex numbers $\mathbf{C}$. A number of things can be extended to mostly arbitrary fields, but we omit this for simplicity.

We will start with a representation U of a group G (which will be a product of general linear groups in this talk) and want to find an interpretation for the orbits of G on U (or at least the well-behaved ones). The plan is to use Borel–Weil to identify U with sections of a vector bundle $\mathcal{U}$ over a Grassmannian (more generally, we should consider partial flag varieties, but most of what we understand happens over Grassmannians). Then we try to find interesting subvarieties in the fibers of $\mathcal{U}$ and glue them together to get a subvariety in the total space of $\mathcal{U}$ and apply the previous sections to get interesting varieties in the Grassmannian, which we might hope classify our orbits. In all cases, the fibers can be interpreted as a smaller representation U' for a smaller group G' such that G' acts on U' with finitely many orbits, and the subvarieties are the closures of these orbits.

3.1 $\mathbf{C}^5 \otimes \bigwedge^2 \mathbf{C}^5$

(This case is in fact easy to handle more directly, but we want to illustrate our approach.)

Let $U = A \otimes \bigwedge^2 B$ with $\dim A = \dim B = 5$ and $G = \mathbf{GL}(A) \times \mathbf{GL}(B)$. For our Grassmannian, we take $\mathbf{P}(A^*) = \mathbf{Gr}(1, A^*)$, and for our bundle, we take $\mathcal{U} = \mathcal{R}^* \otimes \bigwedge^2 B \cong \mathcal{O}(-1) \otimes \bigwedge^2 B$. This almost fits into the setting of the Buchsbaum–Eisenbud complex if we take $E = B^*$, but there is a twist by the line bundle $\mathcal{O}(1)$ here. We can think of this line bundle as fulfilling the role of grading shift. We get the following locally free resolution over $\mathcal{O}_{\mathcal{U}} = \mathrm{Sym}(\bigwedge^2 B^* \otimes \mathcal{O}_{\mathcal{U}}(1))$:

$$0 \to (\det B^*)^{\otimes 2} \otimes \mathcal{O}_{\mathcal{U}}(-5) \to (\det B^*) \otimes B^* \otimes \mathcal{O}_{\mathcal{U}}(-3) \to \bigwedge^4 B^* \otimes \mathcal{O}_{\mathcal{U}}(-2) \to \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{C}} \to 0$$

where $\mathcal{C}$ has codimension 3 in the total space of $\mathcal{U}$. Its singular locus is the zero section of $\mathcal{U}$, and has codimension 10 in $\mathcal{U}$.

Since $\mathcal{U}$ is Cohen–Macaulay (since it is smooth), we can identify the notions of grade and codimension for ideals. Now pick $v \in U$. This gives a section of $\mathcal{U}$, and if it is generic, then $C = \mathcal{C} \cap v(\mathbf{P}(A^*))$ will have codimension 3 in $v(\mathbf{P}(A^*)) \cong \mathbf{P}(A^*)$. If v is very generic, then C will also be smooth.

By generic perfection, we get a locally free resolution for $\mathcal{O}_C$:

$$\begin{aligned} 0 \to (\det B^*)^{\otimes 2} \otimes \mathcal{O}_{\mathbf{P}(A^*)}(-5) \to (\det B^*) \otimes B^* \otimes \mathcal{O}_{\mathbf{P}(A^*)}(-3) \\ \to \bigwedge^4 B^* \otimes \mathcal{O}_{\mathbf{P}(A^*)}(-2) \to \mathcal{O}_{\mathbf{P}(A^*)} \to \mathcal{O}_C \to 0. \end{aligned}$$

This gives enough information to see that $\omega_C = \mathcal{O}_C$, $\dim \mathrm{H}^0(C;\mathcal{O}_C) = 1$, and that $\dim \mathrm{H}^1(C;\mathcal{O}_C) = 1$. In particular, C is a curve of genus 1. We can also deduce that C is projectively normal and embedded by a complete linear series.

Conversely, given a curve C of genus 1 embedded in $\mathbf{P}(A^*)$ by a complete linear series, its homogeneous ideal I is generated by 5 quadrics and is a codimension 3 Gorenstein ideal. The Buchsbaum–Eisenbud classification of such ideals says that we can recover a section $v \in U$ which gives rise to C. Hence we have an isomorphism

$$U^{\mathrm{gen}} /\!\!/ G \xrightarrow{\cong} \{\text{degree 5 smooth curves of genus 1 in } \mathbf{P}^4\}/\cong$$

where the superscript gen just refers to the open subset of U giving rise to smooth curves, and on the right we quotient by some appropriate notion of isomorphism.

What if we change our choice of Grassmannian? One possibility is to choose $\mathbf{Gr}(3, A^*)$ and take $\mathcal{U} = \mathcal{R}^* \otimes \bigwedge^2 B$. The fibers of this bundle are of the form $\mathbf{C}^3 \otimes \bigwedge^2 \mathbf{C}^5$. Under the action of $\mathbf{GL}(3) \times \mathbf{GL}(5)$, it has an invariant degree 15 hypersurface. The method above would produce a degree 5 hypersurface in $\mathbf{Gr}(3, A^*)$. One can check that this is the Chow form of the curve C above.

Another possibility is to take $\mathbf{Gr}(4, A^*) = \mathbf{P}(A)$ and $\mathcal{U} = \mathcal{Q} \otimes \bigwedge^2 B = \bigwedge^3 \mathcal{Q}^* \otimes \mathcal{R}^* \otimes (\det A) \otimes \bigwedge^2 B$. The fibers look like $\mathbf{C}^4 \otimes \bigwedge^2 \mathbf{C}^5$. Under the action of $\mathbf{GL}(4) \times \mathbf{GL}(5)$, it has an invariant degree 40 hypersurface. The method above produces a degree 10 hypersurface in $\mathbf{P}(A)$. One can check that this is the projective dual of the curve C.

3.2 $\bigwedge^3 \mathbf{C}^9$

Let $U = \bigwedge^3 V$ with $\dim V = 9$ and $G = \mathbf{GL}(V)$. For our Grassmannian, we take $\mathbf{P}(V^*) = \mathbf{Gr}(1, V^*)$. For our bundle, we take $\mathcal{U} = \bigwedge^2 \mathcal{Q}^* \otimes \mathcal{R}^*$. Since $\operatorname{rank} \mathcal{Q}^* = 8$ is even, the Buchsbaum–Eisenbud complex doesn't apply.

The fibers of $\mathcal{U}$ are 8×8 skew-symmetric matrices, so the Pfaffian loci give candidates for interesting subvarieties. The 8×8 Pfaffian itself has an easy resolution over $\mathcal{O}_{\mathcal{U}} = \mathrm{Sym}(\bigwedge^2 \mathcal{Q} \otimes \mathcal{O}(-1))$:

$$0 \to (\det \mathcal{Q}) \otimes \mathcal{O}_{\mathcal{U}}(-4) \to \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{Y}} \to 0.$$

We can simplify this by noting that $\det \mathcal{Q} = \mathcal{O}(1)$. Its singular locus is locally cut out by the ideal of 6×6 Pfaffians. This also has a known resolution given by the Józefiak–Pragacz complex (for an arbitrary commutative ring the functors need to be defined differently, see [Pra] for details). If we apply it here, we get

$$\begin{aligned} 0 \to \mathcal{O}_{\mathcal{U}}(-9) \to \bigwedge^2 \mathcal{Q} \otimes \mathcal{O}_{\mathcal{U}}(-7) \to \mathbf{S}_{2,1^6} \mathcal{Q} \otimes \mathcal{O}_{\mathcal{U}}(-7) \\ \to (S^2 \mathcal{Q} \otimes \mathcal{O}_{\mathcal{U}}(-4)) \oplus ((S^2 \mathcal{Q})^* \otimes \mathcal{O}_{\mathcal{U}}(-5)) \\ \to \mathbf{S}_{2,1^6} \mathcal{Q} \otimes \mathcal{O}_{\mathcal{U}}(-4) \to \bigwedge^6 \mathcal{Q} \otimes \mathcal{O}_{\mathcal{U}}(-3) \to \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{X}} \to 0. \end{aligned}$$

The variety $\mathcal{X}$ has codimension 6 and fits into the hypothesis of the generic perfection theorem. Its singular locus is cut out by the 4×4 Pfaffians and has codimension 21.

If we choose a sufficiently generic vector $v \in U$, then $Y = \mathcal{Y} \cap v(\mathbf{P}(V^*))$ and $X = \mathcal{X} \cap v(\mathbf{P}(V^*))$ will have codimension 1 and 6 in $\mathbf{P}(V^*)$, respectively, the singular locus of Y is X, and X is smooth. We note that Y is a cubic hypersurface.

To figure out X, we first note that the resolution shows that $\omega_X = \mathcal{O}_X$. We can use Bott's theorem (which assumes characteristic 0) to show that $\dim \mathrm{H}^0(X; \mathcal{O}_X) = 1$ and $\dim \mathrm{H}^1(X; \mathcal{O}_X) = 2$, and hence we see that X is an Abelian surface (modulo choosing a point to be its origin). Furthermore, we can show that the induced polarization on X is indecomposable and of type $(3, 3)$.

This situation is classical and was studied by Coble (Y is known as the Coble cubic, see [Bea]). We would like to say that all Abelian surfaces embedded in $\mathbf{P}^8$ via a $(3, 3)$ polarization come from this construction. We have not written down a complete proof for this yet.

3.3 $\bigwedge^4 \mathbf{C}^8$

It's not always Pfaffians.

We take $U = \bigwedge^4 V$ with $\dim V = 8$ and $G = \mathbf{GL}(V)$. We take our Grassmannian to be $\mathbf{P}(V^*)$ and $\mathcal{U} = \bigwedge^3 \mathcal{Q}^* \otimes \mathcal{R}^*$. Note that the fibers of $\mathcal{U}$ are $\bigwedge^3 \mathbf{C}^7$. This is not a familiar situation, but it is true that $\mathbf{GL}(7)$ acts on it with finitely many orbits. So we first figure out how to construct the free resolutions for these orbit closures (subject of a different talk).

Then for a sufficiently generic $v \in U$, our construction will yield the Kummer quotient X (with $(2, 2, 2)$-polarization) of a non-hyperelliptic Abelian 3-fold inside of a quartic hypersurface Y, all embedded in $\mathbf{P}(V^*)$. This situation was also studied by Coble, and Y is the Coble quartic.

There are many other representations and interesting examples (including orthogonal and symplectic groups) that we have studied (mostly coming from [Vin]), but I hope this gives a taste of the kinds of information that a systematic use of free resolutions can provide.

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