1 Introduction

Throughout these lectures, \( n \geq 2 \) will be a fixed integer and \( q \) an indeterminate. (Virtually all objects that we shall consider will depend on this \( n \), but we shall not repeat it all the time neither record it explicitly in our notation.)

Using the representation theory of the quantum affine algebra \( U_q(\widehat{\mathfrak{g}_n}) \) one can introduce for each pair of partitions \( \lambda \) and \( \mu \) of the same integer \( m \) two polynomials \( d_{\lambda,\mu}(q) \) and \( e_{\lambda,\mu}(q) \) with integer coefficients. (They depend on \( n \).) For example, for \( n = 2 \) and \( m = 4 \), one has:

\[
\begin{array}{cccccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
(4) & (31) & (2^2) & (21^2) & (1^4) & e_{\lambda,\mu}(q) & (4) & (31) & (2^2) & (21^2) & (1^4) \\
(4) & 1 & 0 & 0 & 0 & 0 & (4) & 1 & q & q^2 & 0 & 0 \\
(31) & q & 1 & 0 & 0 & 0 & (31) & 0 & 1 & q & 0 & q \\
(2^2) & 0 & q & 1 & 0 & 0 & (2^2) & 0 & 0 & 1 & q & q^2 \\
(21^2) & q & q^2 & q & 1 & 0 & (21^2) & 0 & 0 & 0 & 1 & q \\
(1^4) & q^2 & 0 & 0 & q & 1 & (1^4) & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

These polynomials are defined as the coefficients of two canonical bases of the Fock space representation of \( U_q(\widehat{\mathfrak{g}_n}) \). They were studied in [LLT1, LT1] motivated by the work of James on modular representations of symmetric groups and finite general linear groups [J]. It was conjectured in [LLT1, LT1] that the \( d_{\lambda,\mu}(1) \) are some decomposition numbers for Hecke algebras and quantized Schur algebras at roots of unity. These conjectures were proved by Ariki [A] (for Hecke algebras) and by Varagnolo-Vasserot [VV] (for Schur algebras).

The \( e_{\lambda,\mu}(q) \) have been shown to coincide with the coefficients of Lusztig’s character formula for the simple modules of \( U_q(\mathfrak{gl}_r) \), where \( r \geq m \) and \( v^2 \) is a primitive \( n \)th root of 1 [VV]. Similarly, the \( d_{\lambda,\mu}(q) \) are the coefficients of Soergel’s formula for the indecomposable tilting \( U_q(\mathfrak{gl}_r) \)-modules [VV, GW, LT2]. In particular, both families are parabolic Kazhdan-Lusztig polynomials for the Coxeter groups of type \( A_{m-1}^{(1)} \). Recently, some of them have been explicitly described in terms of the Littlewood-Richardson coefficients [LM] (see also [CT]).
The polynomials \( e_{\lambda \mu}(q) \) contain as special cases the Kostka-Foulkes polynomials and more generally the \( q \)-Littlewood-Richardson coefficients of \([\text{CL}, \text{L1T2}, \text{LT2}]\) defined combinatorially in terms of \( n \)-ribbon tableaux.

The aim of these lectures is to review the definition, the calculation and the main properties of \( d_{\lambda \mu}(q) \) and \( e_{\lambda \mu}(q) \). We shall emphasize the close connections between the Fock space representation of \( U_q(\mathfrak{sl}_n) \) and the ring of symmetric functions. In the last section, we shall indicate some connections with the Macdonald polynomials.

2 The affine Lie algebra \( \widehat{\mathfrak{sl}}_n \)

Let \( A = [a_{ij}]_{0 \leq i,j \leq n-1} \) be the \( n \times n \) matrix

\[
\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
: & : & : & \ddots & : & : & : \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
-1 & 0 & 0 & \cdots & 0 & -1 & 2 \\
\end{bmatrix}
\]

\( (n = 2), \quad (n \geq 3). \)

The complex Lie algebra \( \mathfrak{g} \) with generators \( e_i, f_i, h_i \) \((0 \leq i \leq n - 1), d, \) submitted to the relations

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad (1)
\]

\[
[d, h_i] = 0, \quad [d, e_i] = e_0, \quad [d, f_0] = f_0, \quad (2)
\]

\[
[d, e_i] = 0, \quad [d, f_i] = 0, \quad (i \neq 0) \quad (3)
\]

\[
[e_i, f_j] = \delta_{ij} h_i, \quad (4)
\]

\[
(ad e_i)^{1-a_{ij}} e_j = 0, \quad (ad f_i)^{1-a_{ij}} f_j = 0, \quad (i \neq j) \quad (5)
\]

is the Kac-Moody algebra of type \( A_{n-1}^{(1)} \), also denoted \( \widehat{\mathfrak{sl}}_n \) [Ka]. (Here, for \( x \in \mathfrak{g}, ad x \) means the endomorphism \( y \mapsto [x,y] \).) The subalgebra of \( \mathfrak{g} \) obtained by omitting the generator \( d \) is denoted by \( \mathfrak{g}' \). The universal enveloping algebra of \( \mathfrak{g} \) is denoted by \( U(\mathfrak{g}) \).

3 The Fock space representation of \( \widehat{\mathfrak{sl}}_n \)

3.1 A concrete realization of \( \mathfrak{g} \) can be obtained by letting it act on the Fock space. This goes back to [JIMI].

Let \( \mathcal{P} \) be the set of all partitions. By convention \( \mathcal{P} \) contains the empty partition \( \emptyset \). As usual, we identify a partition \( \lambda \) with its Young diagram. A node of \( \lambda \) whose content is congruent to \( i \) modulo \( n \) is called an \( i \)-node.

Let \( \mathcal{F} \) be an infinite-dimensional complex vector space with a distinguished \( \mathbb{C} \)-basis \( \{s(\lambda)\} \) indexed by \( \lambda \in \mathcal{P} \). We define endomorphisms \( e_i, f_i \) \((0 \leq i \leq n - 1), d \) of \( \mathcal{F} \), by

\[
e_i s(\lambda) = \sum_{\mu} s(\mu), \quad f_i s(\lambda) = \sum_{\nu} s(\nu), \quad d s(\lambda) = N_0(\lambda) s(\lambda), \quad (6)
\]

where the first sum is over all partitions \( \mu \) obtained by removing from \( \lambda \) an \( i \)-node, the second sum is over all partitions \( \nu \) obtained from \( \lambda \) by adding an \( i \)-node, and \( N_0(\lambda) \) denotes the number of
0-nodes of \( \lambda \). It is easy to see that \( e_i \mapsto e_i, f_i \mapsto f_i, \) and \( d \mapsto d \) defines a representation of \( g \), that is, setting \( h_i := [e_i, f_i] \), one can check that \( e_i, f_i, h_i, d \) satisfy relations (1) (2) (3) (4) (5). In the sequel we shall write for simplicity \( e_i, f_i, h_i, d \) in place of \( e_i, f_i, h_i, d \).

**Example 1** Let \( n = 2 \). We have (see Figure 1)

\[
\begin{align*}
&f_0 s(3, 1) = s(3, 2) + s(3, 1, 1), \\
&f_1 s(3, 1) = s(4, 1), \\
&e_0 s(3, 1) = s(2, 1), \\
&e_1 s(3, 1) = s(3).
\end{align*}
\]

By construction, \( s(\emptyset) \) is a highest weight vector, that is, a common eigenvector of \( h_0, \ldots, h_{n-1} \), annihilated by all \( e_i \)'s. The subrepresentation \( U(g) s(\emptyset) \) generated by \( s(\emptyset) \) is irreducible. It is the simplest infinite-dimensional representation of \( g \), and for this reason it is often called the basic representation. One may regard it as an affine analogue of the vector representation of the finite-dimensional Lie algebra \( \mathfrak{sl}_n \).

### 3.2

Let \( \text{Sym} \) denote the \( \mathbb{C} \)-algebra of symmetric functions in an infinite set of variables \([\text{Med}]\). For \( \lambda \in \mathcal{P} \), let \( s_\lambda \) be the Schur function and \( p_\lambda \) the product of power sums corresponding to \( \lambda \). We denote by \( \langle \cdot, \cdot \rangle \) the scalar product for which the basis \( \{s_\lambda\} \) is orthonormal. To \( f \in \text{Sym} \) we associate the endomorphism \( D_f \) defined as the adjoint of the operator of multiplication by \( f \), that is,

\[
\langle D_f g, h \rangle = \langle g, fh \rangle \quad (f, g, h \in \text{Sym}).
\]

We have a natural isomorphism of vector spaces from \( \text{Sym} \) to \( \mathcal{F} \) given by \( s_\lambda \mapsto s(\lambda) \). From now on, we shall identify these two spaces via this isomorphism. For \( k \in \mathbb{Z}^* \), let \( b_k \in \text{End} \mathcal{F} \) be defined by

\[
b_k = \begin{cases} 
D_{pk} & \text{if } k > 0, \\
\text{multiplication by } p_{|k|} & \text{if } k < 0.
\end{cases}
\]

One can show that the operators \( b_{nk} (k \in \mathbb{Z}^*) \) commute with the action of \( g^\prime \). This implies that for any \( \lambda \in \mathcal{P} \), the power sum \( p_{n\lambda} \) is a highest weight vector of \( \mathcal{F} \). On the other hand, the \( b_k \) with \( k \) not divisible by \( n \) belong to the image of \( g \) in \( \text{End} \mathcal{F} \). It follows that as a \( g \)-module, \( \mathcal{F} \) decomposes into the direct sum

\[
\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}} U(g) p_{n\lambda} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}[p_k ; k \neq 0 \mod n] p_{n\lambda}.
\]

Here, all summands are irreducible, and isomorphic as \( g^\prime \)-modules. Moreover, the degree generator \( d \) acts on the highest weight vectors by

\[
d p_{n\lambda} = |\lambda| p_{n\lambda}.
\]

Hence it separates the irreducible components generated by power sums of different degrees. Note however that at this stage there is no canonical choice of highest weight vectors in a given degree, and in fact the basis \( \{p_{n\lambda} \mid \lambda \in \mathcal{P} \} \) of the space of highest weight vectors could well be replaced by any \( \mathbb{C} \)-basis of \( \mathbb{C}[p_{nk} ; k \in \mathbb{N}^*] \) consisting of homogeneous elements. We are now going to see that by \( q \)-deforming this picture, we are lead to a distinguished basis of highest weight vectors.
4 The quantum affine algebra $U_q(\hat{\mathfrak{sl}}_n)$

Let $\mathbb{K} = \mathbb{C}(q)$. The algebra $U_q = U_q(\hat{\mathfrak{sl}}_n)$ is defined [Dr, Ji] as the associative algebra over $\mathbb{K}$ with generators $E_i, F_i, K_i, K_i^{-1}$ ($0 \leq i \leq n - 1$), $D, D^{-1}$ submitted to the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,$$

$$DD^{-1} = D^{-1} D = 1, \quad DK_i = K_i D,$$

$$DE_0 D^{-1} = q^{-1} E_0, \quad DF_0 D^{-1} = q F_0,$$

$$DE_i D^{-1} = 0, \quad DF_i D^{-1} = 0, \quad (i \neq 0),$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^{1-a_{ij} - k} E_j E_i^k = 0, \quad (i \neq j),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^{1-a_{ij} - k} F_j F_i^k = 0, \quad (i \neq j).$$

Here, we have used the $q$-integers and $q$-binomial coefficients given by

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]! = [k] [k - 1] \cdots [1], \quad \binom{m}{k} = \frac{[m]!}{[m-k]! [k]!}.$$

The subalgebra obtained by omitting the generator $D$ is denoted by $U'_q$.

5 The Fock space representation of $U_q(\hat{\mathfrak{sl}}_n)$

Let $\mathcal{F}_q = \mathcal{F} \otimes \mathbb{K}$ be the Fock space over the field $\mathbb{K}$. Following [H, MM] one has an action of $U_q$ on $\mathcal{F}_q$ defined as follows.

Let $\lambda$ and $\mu$ be two Young diagrams such that $\mu$ is obtained from $\lambda$ by adding an $i$-node $\gamma$. Such a node is called a removable $i$-node of $\mu$, or an indent $i$-node of $\lambda$. Let $I^\nu_i (\lambda, \mu)$ (resp. $R^\nu_i (\lambda, \mu)$) be the number of indent $i$-nodes of $\lambda$ (resp. of removable $i$-nodes of $\lambda$) situated to the right of $\gamma$ ($\gamma$ not included). Set $N^\nu_i (\lambda, \mu) = I^\nu_i (\lambda, \mu) - R^\nu_i (\lambda, \mu)$. Then

$$F_i s(\lambda) = \sum_{\mu} q^{N^\nu_i(\lambda, \mu)} s(\mu),$$

where the sum is over all partitions $\mu$ such that $\mu/\lambda$ is an $i$-node. Similarly

$$E_i s(\mu) = \sum_{\lambda} q^{-N^\nu_i(\lambda, \mu)} s(\lambda),$$

where the sum is over all partitions $\lambda$ such that $\mu/\lambda$ is an $i$-node, and $N^\nu_i(\lambda, \mu)$ is defined as $N^\nu_i(\lambda, \mu)$ but replacing right by left. Finally,

$$D s(\lambda) = q^{N_0(\lambda)} s(\lambda).$$

These equations make $\mathcal{F}_q$ into an integrable representation of $U_q$. 

4
Example 2 Let \( n = 2 \). We have (see Figure 1)
\[
F_0 \, s(3, 1) = q^{-1} \, s(3, 2) + s(3, 1, 1), \quad F_1 \, s(3, 1) = s(4, 1),
\]
\[
E_0 \, s(3, 1) = q^{-2} \, s(2, 1), \quad E_1 \, s(3, 1) = s(3).
\]
As in the classical case, the subrepresentation \( U_q \theta \) is the simplest highest weight irreducible representation and is called the basic representation of \( U_q \).

In order to work out the decomposition of \( F_q \) into irreducible components, Kashiwara, Miwa and Stern have introduced some \( q \)-analogues \( B_k \) of the operators \( b_{nk} \) [KMS]. (Apparently, there is no natural \( q \)-analogue of \( b_k \) for \( k \) non divisible by \( n \).) Unfortunately, the action of these \( q \)-bosons is difficult to describe combinatorially. To get around this problem, a different family of operators has been defined in [LLT2], whose action on \( F_q \) has a simple expression in terms of ribbons.

![Figure 2: An 11-ribbon of height \( h(R) = 6 \)](image)

![Figure 3: A skew diagram \( \theta \) with its subdiagram \( \theta \downarrow \) shaded](image)

![Figure 4: A horizontal 5-ribbon strip of weight 4 and spin 7](image)

A ribbon is a connected skew Young diagram \( R \) of width 1, i.e. which does not contain any \( 2 \times 2 \) square (see Figure 2). The rightmost and bottommost cell is called the origin of the ribbon. An \( n \)-ribbon is a ribbon made of \( n \) square cells. Let \( \theta \) be a skew Young diagram, and let \( \theta \downarrow \) be the horizontal strip made of the bottom cells of the columns of \( \theta \) (see Figure 3). We say that \( \theta \) is a horizontal \( n \)-ribbon strip of weight \( k \) if it can be tiled by \( k \) \( n \)-ribbons the origins of which lie in \( \theta \downarrow \). One can check that if such a tiling exists, it is unique (see Figure 4). Define the spin of \( R \) as
\[
\text{spin}(R) := h(R) - 1
\]
Figure 5: Calculation of $v_{(2)}$ and $v_{(1,1)}$ for $n = 2$

where $h(R)$ is the height of $R$, and the spin of an $n$-ribbon strip as the sum of the spins of the ribbons which tile it.

For $k \in \mathbb{N}^*$, let $V_k$ be the linear operator acting on $\mathcal{F}$ by

$$V_k s(\lambda) = \sum_{\mu} (-q)^{-\text{spin}(\mu/\lambda)} s(\mu),$$

(18)

the sum being over all $\mu$ such that $\mu / \lambda$ is a horizontal $n$-ribbon strip of weight $k$. These operators pairwise commute and they also commute with the action of the subalgebra $U_q' [\text{LLT2, LT2}]$. Hence for each $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}$, the vector

$$v_\lambda := V_{\lambda_1} \cdots V_{\lambda_r} s(\emptyset)$$

(19)

is a highest weight vector of the $U_q'$-module $\mathcal{F}_q$, and one has the decomposition

$$\mathcal{F}_q = \bigoplus_{\lambda \in \mathcal{P}} U_q v_\lambda$$

(20)

where all summands are irreducible, and isomorphic as $U_q'$-modules.

Note that it follows easily from (18) that at $q = 1$,

$$V_k s_\lambda = p_n [h_k] s_\lambda,$$

that is, $V_k$ is a $q$-analogue of the multiplication by the plethysm $p_n[h_k] = h_k(x_1^n, x_2^n, \ldots)$. Hence $v_\lambda$ is a $q$-analogue of the symmetric function $p_n[h_\lambda]$.

**Example 3** Let $n = 2$. The highest weight vectors in degree 4 are as follows (see Figure 5)

$$v_{(2)} = V_2 s(\emptyset) = s(4) - q^{-1} s(3,1) + q^{-2} s(2,2),$$

$$v_{(1,1)} = V_1^2 s(\emptyset) = s(4) - q^{-1} s(3,1) + (q^{-2} + 1) s(2,2) - q^{-1} s(2,1,1) + q^{-2} s(1,1,1,1).$$

**6 The bar involution of $\mathcal{F}_q$**

6.1 There is a unique map $x \mapsto \overline{x}$ from $\mathcal{F}_q$ to itself satisfying

$$\overline{s(\emptyset)} = s(\emptyset),$$

(21)

$$\overline{\varphi(q)x + \psi(q)y} = \varphi(q^{-1})\overline{x} + \psi(q^{-1})\overline{y}, \quad (x, y \in \mathcal{F}_q, \varphi, \psi \in \mathcal{K}),$$

(22)

$$\overline{F_i x} = F_i \overline{x}, \quad (0 \leq i \leq n - 1, x \in \mathcal{F}_q),$$

(23)
Indeed, by (19) (20) (21) (22) and (24) this map is determined on the space of highest weight vectors of $\mathcal{F}_q$, and by (23) it can be uniquely extended to the whole space $\mathcal{F}_q$. Write

$$s(\mu) = \sum_{\lambda} a_{\lambda,\mu}(q) s(\lambda), \quad A_m = [a_{\lambda,\mu}(q)]_{|\lambda|=|\mu|=m}.$$

**Theorem 4 ([LT1, LT2])** The matrix $A_m$ has entries in $\mathbb{Z}[q,q^{-1}]$ and is lower unitriangular if its rows and columns are labelled by partitions arranged in decreasing lexicographic order. More precisely, $a_{\lambda,\mu}(q)$ can be nonzero only if $\lambda$ is less or equal to $\mu$ for the dominance order on partitions.

The triangularity property is not obvious. To prove it, one uses the ‘fermionic’ realization of $\mathcal{F}_q$ given in [KMS] to obtain a description of the bar-involution in terms of the straightening rules of $q$-wedge products. However, this other description is not appropriate for practical calculations. We shall now explain a different algorithm.

### 6.2 A partition $\lambda$ is said to be $n$-regular if it has no part repeated more than $n-1$ times, that is, if we write $\lambda$ in multiplicative form $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ all $m_i$’s are strictly less than $n$. We shall use the ladder decomposition of an $n$-regular partition ([JK], 6.3.51, p.283). The ladders of $\lambda$ are the intersections of its Young diagram with the straight lines of equation

$$y = (1-n)x + k, \quad (k = 0, 1, 2, \ldots).$$

(Here we take the origin of coordinates to be the center of the leftmost bottom box of $\lambda$). By construction, the nodes of $\lambda$ lying on a given ladder $y = (1-n)x + k$ all have the same residue $n-k \mod n$. Let $s$ be the number of ladders containing at least one node of $\lambda$. Denote these ladders from left to right by $L_1, \ldots, L_s$, so that $L_1$ is the ladder through the origin and $L_s$ is the rightmost ladder intersecting $\lambda$. Let $k_i$ be the number of nodes of $\lambda$ lying on $L_i$, and let $r_i$ be their common $n$-residue. We define

$$t(\lambda) = F^{(k_1)}_{r_1} F^{(k_{r-1})}_{r_{r-1}} \cdots F^{(k_1)}_{r_1} s(\emptyset),$$

where $F^{[r]}_i = F^{(r)}_i / |r|!.$

**Example 5** Let $n = 3$ and $\lambda = (3, 3, 2)$. The ladder decomposition of $\lambda$ is shown in Figure 6. This yields

$$t(3, 3, 2) = F_1 F_2^{(2)} F_0 F_1^{(2)} F_2 F_0 s(\emptyset) = s(3, 3, 2) + q s(3, 1, 1, 1, 1).$$

Figure 6: The ladder decomposition of $(3, 3, 2)$ for $n = 3$.
More generally, if \( \lambda = (1^{m_1}, \ldots, r^{m_r}) \) is not \( n \)-regular we write \( \alpha = (1^{a_1}, \ldots, r^{a_r}) \) and \( \beta = (1^{b_1}, \ldots, r^{b_r}) \) where \( m_i = na_i + b_i \) and \( 0 \leq b_i < n \). Then \( \beta \) is \( n \)-regular and has a ladder decomposition \( L_1, \ldots, L_s \) with corresponding integers \( r_1, \ldots, r_s, k_1, \ldots, k_s \). We set

\[
t(\lambda) = F_r^{(k_s)} F_{r_{s-1}}^{(k_{s-1})} \cdots F_{r_1}^{(k_1)} v_\alpha,
\]

where \( v_\alpha \) is given by (19).

**Example 6** Let \( n = 2 \). The vectors \( t(\lambda) \) for all partitions \( \lambda \) of 4 are

\[
\begin{align*}
t(4) &= F_1 F_0 F_1 F_0 s(0) = s(4) + q s(3, 1) + q s(2, 1, 1) + q^2 s(1, 1, 1, 1), \\
t(3, 1) &= F_0 F_1^{(2)} F_0 s(0) = s(3, 1) + q s(2, 2) + q^2 s(2, 1, 1), \\
t(2, 2) &= v_{(2)} = s(4) - q^{-1} s(3, 1) + q^{-2} s(2, 2), \\
t(2, 1, 1) &= F_1 F_0 v_{(1)} = s(4) + q s(3, 1) - q^{-1} s(2, 1, 1) - s(1, 1, 1, 1), \\
t(1, 1, 1, 1) &= v_{(1, 1)} = s(4) - q^{-1} s(3, 1) + (q^{-2} + 1) s(2, 2) - q^{-1} s(2, 1, 1) + q^{-2} s(1, 1, 1, 1).
\end{align*}
\]

Because of the combinatorial simplicity of (15) and (18), the expansion of \( t(\mu) \) on the basis \( \{s(\lambda)\} \) is easily computable. Clearly, it only involves vectors \( s(\lambda) \) with \( |\lambda| = |\mu| \). Write

\[
t(\mu) = \sum_{\lambda} t_{\lambda, \mu}(q) s(\lambda), \quad T_m = [t_{\lambda, \mu}(q)]_{|\lambda| = |\mu| = m}.
\]

By construction \( t_{\lambda, \mu}(q) \in \mathbb{Z}[q, q^{-1}] \).

**Lemma 7** \( \{t(\lambda), \lambda \in \mathcal{P}\} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{F}_q \) invariant under the bar-involution \( x \mapsto \overline{x} \).

The fact that \( t(\lambda) \) is bar-invariant follows immediately from (23) and (24). The proof that the \( t(\lambda) \)'s with \( n \)-regular \( \lambda \) form a basis of the basic representation was given in [LLT1]. Since the \( V_k \)'s and the \( F_i \)'s commute with each other, we see that the \( t(\lambda) \)'s with the same \( n \)-singular part \( \alpha \) form a \( \mathbb{Z} \)-basis of the subrepresentation \( U_q v_\alpha \), and the result follows.

The bar-invariance of \( t(\lambda) \) yields

\[
A_m = T_m (\overline{T}_m)^{-1},
\]

where \( \overline{T}_m = [t_{\lambda, \mu}(q^{-1})]_{|\lambda| = |\mu| = m} \). Hence, from the calculation of the \( t(\lambda) \)'s we deduce the matrix of the bar-involution. Note however that this involves inverting the matrix \( T_m \) of size the number of partitions of \( n \). This is the time and space consuming step of the algorithm. Note also that \( T_m \) is not triangular, so using this approach, the triangularity of \( A_m \) remains rather mysterious.

**Example 8** Let us calculate \( A_4 \) for \( n = 2 \). The partitions of 4 are arranged in the order

\((4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\).

From Example 6 we get

\[
T_4 = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
q & 1 & -q^{-1} & q & -q^{-1} \\
0 & q & q^{-2} & 0 & q^{-2} + 1 \\
q & q^2 & 0 & -q^{-1} & -q^{-1} \\
q^2 & 0 & 0 & 1 & q^{-2}
\end{bmatrix}.
\]
It follows that

\[
A_4 = T_4 (T_4)^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
q^{-1} & 1 & 0 & 0 & 0 \\
q^{-2} - 1 & q^{-1} & 1 & 0 & 0 \\
0 & q^{-2} - 1 & q^{-1} & 1 & 0 \\
q^{-2} - 1 & 0 & q^{-2} - 1 & q^{-1} & 1
\end{bmatrix}.
\]

7 The canonical bases of $\mathcal{F}_q$

7.1 Let $L$ (resp. $L^-$) be the free $\mathbb{Z}[q]$-module (resp. the free $\mathbb{Z}[q^{-1}]$-module) with basis \{s(\lambda)\}.

Theorem 9 There exist two bases $B = \{ G(\lambda) \mid \lambda \in \mathcal{P} \}$ and $B^- = \{ G^-(\lambda) \mid \lambda \in \mathcal{P} \}$ of $\mathcal{F}_q$ characterized by the following properties

\[
\overline{G(\lambda)} = G(\lambda), \quad G(\lambda) \equiv s(\lambda) \mod qL, \quad G^-(\lambda) \equiv s(\lambda) \mod q^{-1}L^-.
\]

Proof — Let us prove the existence of $B$. Fix $m \in \mathbb{N}$ and let $\lambda^1 > \lambda^2 > \cdots > \lambda^k$ be the list of partitions of $m$ arranged in decreasing lexicographic order. By Theorem 4, $s(\lambda^k) = s(\lambda^k)$, so we can take $G(\lambda^k) = s(\lambda^k)$. We now argue by induction and suppose that for a certain $r < k$ we have constructed vectors $G(\lambda^{r+1}), G(\lambda^{r+2}), \ldots, G(\lambda^k)$ satisfying the conditions of the theorem. Moreover, we assume that

\[
G(\lambda^{r+i}) = s(\lambda^{r+i}) + \sum_{i < j \leq k-r} \alpha_{ij}(q) s(\lambda^{r+j}), \quad (i = 1, \ldots, k-r), \quad (27)
\]

that is, the expansion of $G(\lambda^{r+i})$ only involves vectors $s(\mu)$ with $\mu \leq \lambda^{r+i}$. We can therefore write, by solving a linear system with unitriangular matrix,

\[
\overline{s(\lambda^r)} = s(\lambda^r) + \sum_{1 \leq j \leq k-r} \beta_j(q) G(\lambda^{r+j}),
\]

where the coefficients $\beta_j(q) \in \mathbb{Z}[q, q^{-1}]$. By applying the bar involution to this equation we get that $\beta_j(q^{-1}) = -\beta_j(q)$, hence $\beta_j(q) = \gamma_j(q) - \gamma_j(q^{-1})$ with $\gamma_j(q) \in q\mathbb{Z}[q]$. Now set

\[
G(\lambda^r) = s(\lambda^r) + \sum_{1 \leq j \leq k-r} \gamma_j(q) G(\lambda^{r+j}).
\]

We have $G(\lambda^r) \equiv s(\lambda^r) \mod qL$, $\overline{G(\lambda^r)} = G(\lambda^r)$, and the $s(\mu)$-expansion of $G(\lambda^r)$ is of the form (27) as required, hence the existence of $B$ follows by induction.

The proof of the existence of $B^-$ is similar.

To prove unicity, we show that if $x \in qL$ is bar invariant then $x = 0$. Otherwise write $x = \sum \theta_\lambda(q) s(\lambda)$, and let $\mu$ be the largest partition for which $\theta_\mu(q) \neq 0$. Then $s(\mu)$ occurs in $\mathcal{P}$ with coefficient $\theta_\mu(q^{-1})$, hence $\theta_\mu(q) = \theta_\mu(q^{-1})$. But since $\theta_\mu(q) \in q\mathbb{Z}[q]$ this is impossible. □

It is easy to see that the $G(\lambda)$ which belong to the basic representation are precisely those for which $\lambda$ is $n$-regular, and that they coincide with Kashiwara’s lower global base of this irreducible module. In fact, $B$ is a lower global base of $\mathcal{F}_q$ (see e.g. [LM]). Uglov [U] has generalized this construction to the higher level Fock space representations of $U_q$, thereby giving a simple algorithm for computing the canonical basis of any integrable simple $U_q$-module. Recently, Kashiwara [K] has obtained in a similar way a global base for all the $q$-deformed Fock spaces of [KMPY] associated with a perfect crystal of an affine Lie algebra.
7.2 We can now define $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ by
\[ G(\mu) = \sum_{\lambda} d_{\lambda,\mu}(q) s(\lambda), \]  
and
\[ G^{-}(\lambda) = \sum_{\mu} e_{\lambda,\mu}(-q^{-1}) s(\mu). \]  
Note that the proof of Theorem 9 shows how to compute the $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ once the $a_{\lambda,\mu}(q)$ are known. This only involves solving a unitriangular system, so it is rather cheap.

Example 10 Let $n = 2$. Using the matrix $A_4$ computed in Example 8 it is easy to calculate the matrices of $B$ and $B^-$ given in the introduction.

By construction $d_{\lambda,\mu}(q) = e_{\lambda,\mu}(q) = 1$, and $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ belong to $q\mathbb{Z}[q]$ if $\lambda \neq \mu$. Recall that to $\lambda \in \mathcal{P}$ is associated an $n$-core $\lambda_{[n]} \in \mathcal{P}$ and a $n$-quotient $(\lambda^{(0)}, \ldots, \lambda^{(n-1)}) \in \mathcal{P}^n$ (see [JK] or [Med] I.1, ex. 8). They satisfy $|\lambda| = |\lambda_{[n]}| + n \sum_{i \in [n-1]} |\lambda^{(i)}|$, and one can reconstruct $\lambda$ from its $n$-core and $n$-quotient.

Proposition 11 (i) $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ can be nonzero only if $|\lambda| = |\mu|$ and $\lambda_{[n]} = \mu_{[n]}$.
(ii) $d_{\lambda,\mu}(q)$ can be nonzero only if $\lambda$ is less or equal to $\mu$ for the dominance order on $\mathcal{P}$.
(iii) $e_{\lambda,\mu}(q)$ can be nonzero only if $\mu$ is less or equal to $\lambda$ for the dominance order on $\mathcal{P}$.

The polynomials $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ enjoy a less obvious duality property. Write
\[ D_m = [d_{\lambda,\mu}(q)]_{|\lambda| = |\mu'| = m}, \quad J_m = [e_{\lambda,\mu'}(-q)]_{|\lambda| = |\mu'| = m}, \]  
where $\lambda'$ denotes the partition conjugate to $\lambda$.

Theorem 12 ([LT1, LT2]) For all $m \in \mathbb{N}$, one has $D_m = J_m^{-1}$.

The proof relies on the following symmetry of the bar involution
\[ a_{\lambda\mu}(q) = a_{\mu\lambda}(q) \quad (\lambda, \mu \in \mathcal{P}). \]  
(30)

Theorem 12 has been generalized by Uglov to the higher level Fock spaces [U].

8 Relations with the Kazhdan-Lusztig polynomials

8.1 In this section, we fix $r \in \mathbb{N}$ and we consider only partitions $\lambda$ with at most $r$ parts. We can regard them as elements of $\mathbb{Z}^r$ by adding at the end an appropriate string of 0’s. The symmetric group $S_r$ acts on $\mathbb{Z}^r$ by permuting coordinates. We denote by $\hat{S}_r$ the group of transformations of $\mathbb{Z}^r$ generated by $S_r$ and the group of translations with respect to the sublattice $n\mathbb{Z}^r$. We choose as fundamental domain for this action of $\hat{S}_r$ on $\mathbb{Z}^r$ the set
\[ \mathcal{A} = \{ a = (a_1, \ldots, a_r) \in \mathbb{Z}^r \mid -n < a_1 \leq a_2 \leq \cdots \leq a_r \leq 0 \}. \]  
The group $\hat{S}_r$ is isomorphic to the (extended) affine Weyl group of type $A_r^{(1)}$. Therefore, it is endowed with a length function $w \mapsto \ell(w)$ and a Bruhat order $\leq$. For each $u \in \mathbb{Z}^r$ there is a
unique \( w_u \in \hat{S}_r \) of minimal length such that \( w_u^{-1}(u) \in A \). Given a pair \((x, w)\) of elements of \( \hat{S}_r \), with \( x \leq w \), we also have a Kazhdan-Lusztig polynomial \( P_{x, w}(q) \) defined as the coefficient of \( T_x \) in the expansion of the Kazhdan-Lusztig base element \( C_w^u \) of the Hecke algebra associated with \( \hat{S}_r \). (Here we follow Soergel’s convention [So] and normalize the generators \( T_i \) of the Hecke algebra so that \((T_i + q)(T_i - q^{-1}) = 0 \) and therefore \( C_i = T_i + q \).)

Define \( \rho = (r-1, r-2, \ldots, 1, 0) \) and consider two partitions \( \lambda \) and \( \mu \) such that \( |\lambda| = |\mu| \) and \( \lambda_{(n)} = \mu_{(n)} \). This implies that \( u = \lambda + \rho \) and \( v = \mu + \rho \) belong to the same \( \hat{S}_r \)-orbit. Let \( a \) be the intersection of \( A \) with this orbit, and let \( \hat{S}(a) \) be the subgroup of \( \hat{S}_r \) consisting of the \( w \) such that \( w(a) = a \). This is in fact a Young subgroup of \( \hat{S}_r \), and we denote by \( w_0^a \) its longest element. Put \( \hat{a} = w_0^a u \), where \( w_0^a \) is the longest element of \( \hat{S}_r \), and similarly \( \hat{v} = w_0^aw \). Finally, set \( \hat{\omega}_u = w_\hat{a} w_0^a \) and \( \hat{\omega}_v = w_\hat{v} w_0^a \).

**Theorem 13 ([VV])** With the above notation, we have

\[
\epsilon_{\lambda, \mu}(q) = \sum_{x \in \hat{S}(a)} (-q)^{\ell(x)} P_{w_u, w_v}(q),
\]

and

\[
d_{\lambda, \mu}(q) = \sum_{y \in \hat{S}(a)} (-q)^{\ell(y)} P_{w_\hat{\omega}_u, w_{\hat{\omega}}}(q).
\]

The proof is based on the \( q \)-wedge realization of \( \mathcal{F}_q \) and on the quantum affine Schur-Weyl duality of Cherednik, Ginzburg-Vasserot and Chari-Pressley. Uglow has obtained similar results for the higher level Fock spaces [U]. A formula equivalent to (32) was also found independently by Goodman and Wenzl [GW] in the case where \( \mu \) is an \( n \)-regular partition.

The theorem shows that both \( d_{\lambda, \mu}(q) \) and \( e_{\lambda, \mu}(q) \) are parabolic Kazhdan-Lusztig polynomials, as defined by Deodhar [D]. Using the geometric interpretation of such polynomials in terms of Schubert varieties of finite codimension in some affine flag manifold [KT], one obtains

**Corollary 14** The polynomials \( d_{\lambda, \mu}(q) \) and \( e_{\lambda, \mu}(q) \) have non-negative coefficients.

In [VV], Varagnolo and Vasserot have constructed a basis \( B \) of the Fock space by projecting the canonical basis of the Hall algebra of the cyclic quiver of type \( A_1^{(1)} \), and they conjectured that \( B = B \). This conjecture has been proved by Schiffmann [Sc], and it provides another proof of the positivity of \( d_{\lambda, \mu}(q) \).

### 8.2

As an application of Theorem 13, we can deduce some interesting relations between the polynomials \( d_{\lambda, \mu}(q) \). Given two partitions \( \lambda \) and \( \mu \) of \( m \) as in 8.1, we define two partitions \( \hat{\mu} \) and \( \hat{\lambda} \) of \( m' = m + (n-1)r(r-1) \) in the following way. There is a unique decomposition \( \mu = n\alpha + \beta \) with the partition \( \beta' \) being \( n \)-regular. Put

\[
\hat{\mu} = 2(n-1)\rho + w_0(\beta) + n\alpha,
\]

\[
\hat{\lambda} = \lambda + ((n-1)(r-1), \ldots, (n-1)(r-1)).
\]

It is easy to check that the parts of \( \hat{\mu} \) are pairwise distinct, and thus \( \hat{\mu} \) is always \( n \)-regular.

**Theorem 15 ([L])** With the notation above, we have

\[
d_{\hat{\lambda}, \hat{\mu}}(q) = q^{\ell(w_0)} \hat{d}_{\hat{\lambda}, \hat{\mu}}(q^{-1}).
\]

The proof uses Theorem 12 and a theorem of Soergel on Kazhdan-Lusztig polynomials [So].
Example 16 Let \( n = 3 \) and \( \mu = (6, 2, 1) \). The non-zero \( d_{\lambda, \mu} (q) \) are obtained for
\[
\lambda = (6, 2, 1), \ (7, 1, 1), \ (6, 3), \ (8, 1).
\]

We can take \( r = 3 \), so that \( m' = 9 + 2 \cdot 3 = 21 \). We have
\[
\mu = (3, 2, 1) + 3(1,0,0), \quad \hat{\mu} = (1, 2, 3) + 4(2,1,0) + 3(1,0,0) = (12, 6, 3).
\]

Moreover, \( \mu + \rho_r = (8, 3, 1) \) is regular, hence \( \ell(w^0_\mu) = 0 \) in this case. Therefore, taking
\[
\tilde{\lambda} = (10, 6, 5), \ (11, 5, 5), \ (10, 7, 4), \ (12, 5, 4),
\]
respectively, we have
\[
d_{\lambda, (3,2,1,1,1,1)} (q) = q^3 d_{\tilde{\lambda}, (12,6,3)} (q^{-1}).
\]

This formula shows that any vector \( G(\mu') \) of \( B \) can be easily computed from the corresponding vector \( G(\hat{\mu}) \), which belongs to the basic representation of \( U_q \) because \( \hat{\mu} \) is \( n \)-regular. Hence, in a sense, Kashiwara’s global basis of this irreducible submodule contains all the information about the whole basis \( B \) of \( \mathcal{F}_q \).

By taking \( q = 1 \), we see that the decomposition matrix for the quantized Schur algebra of rank \( m \) is a submatrix of the decomposition matrix for the Hecke algebra of rank \( m' \), a result which can also be checked directly (see [L]). This is a kind of ‘quantization’ of a result of Erdmann [E] about symmetric groups and general linear groups.

9 Relations with the Kostka-Foulkes polynomials and some \( q \)-analogues of the Littlewood-Richardson coefficients

9.1 Let \( \lambda \) and \( \mu \) be two partitions of \( m \) with at most \( r \) parts. Lusztig has shown that the Kostka-Foulkes polynomial \( K_{\lambda, \mu}(q) \) is a Kazhdan-Lusztig polynomial of type \( A_{r-1}^{(1)} \). With our notation, this result can be stated as

Theorem 17 ([Lu1, Lu2]) If \( n \geq r \) then \( e_{n\lambda, n\mu}(q) = K_{\lambda, \mu}(q^2) \).

Note that \( n \lambda + \rho \) and \( n \mu + \rho \) belong to the \( \widehat{S}_r \)-orbit of \( \rho \), which is regular if \( n \geq r \). Therefore the sum in (31) reduces to the single term \( P_{w_\mu \mu_\mu, \rho_\lambda \rho_\lambda + \rho}(q) \).

More generally, one can study the polynomials \( e_{n\lambda, \mu}(q) \) for any \( \mu \) and remove the restriction \( n \geq r \). This leads to some \( q \)-analogues of the Littlewood-Richardson coefficients.

Recall the operator \( V_{\mu} : = V_{\mu_1} \cdots V_{\mu_r} \). We have seen that at \( q = 1 \), \( V_{\mu} s(\alpha) = p_n[h_{\mu}] s_{\alpha} \). Write \( s_{\lambda} = \sum_\nu \kappa_{\lambda, \nu} h_{\nu} \), where the \( \kappa_{\lambda, \nu} \) are the inverse Kostka numbers. We introduce the operators \( S_{\lambda} \in \text{End}_{U_q} \mathcal{F}_q \) defined by
\[
S_{\lambda} = \sum_\nu \kappa_{\lambda, \nu} V_{\nu}. \tag{36}
\]

At \( q = 1 \), \( S_{\lambda} \) is the operator of multiplication by \( p_n[s_{\lambda}] \).

By Theorem 15, the vectors \( G(\mu) \) of \( B \) can be easily computed in terms of those \( G(\nu) \) for which \( \nu \) is \( n \)-regular. The following theorem, which may be regarded as a \( q \)-analogue of the Steinberg-Lusztig tensor product theorem for \( U_{\mu}(s_{\nu}) \), shows that similarly the vectors \( G^-(\alpha) \) of \( B^- \) can be obtained in a simple way from those \( G^-(\beta) \) for which the conjugate \( \beta' \) of \( \beta \) is \( n \)-regular.
By (36) and (37) we obtain

\[ /D2 \]

be the polynomial obtained by enumerating the sum of the spins of its ribbons. Let \( /B4 \) and if \( /D2 \) vectors of \( /BZ \) Hence we see that the vectors \( /C4 /B5 /BP \). Then \( /AM \) and weight \( /CP /CQ /AL /BN /AM \). Graphically, \( /B4 /D7 /A1 /A1 /A1 /AQ \) may be described by \( /C4 /D7 /B5 /BM /BP \) and spin \( /AL /BN /AM \). It follows immediately from (18) and (19) that

\[ /B4 /BZ /AC /D2 /AL /BN /AM \]

-ribbon tableaux of shape \( /BZ \). The spin of a ribbon tableau \( /B5 /BP \). We denote by \( /CT \) the set of \( /D2 \)-ribbon tableaux of shape \( /BZ \) and weight \( /B5 /BP \). Figure 7: A 4-ribbon tableau of shape \( /B5 /BP \), weight \( /C4 /D7 /B5 /BM /BP \) and \( /C4 /D7 /B5 /BM /BP \). Theorem 18 ([LT1, LT2]) Let \( \alpha \) be a partition such that \( \alpha' \) is \( n \)-singular. Write \( \alpha = \beta + n\lambda \) where \( \beta' \) is \( n \)-regular. Then \( G^-(\alpha) = S_{\lambda} G^-(\beta). \)

In particular, taking \( \alpha = \emptyset \) we get

\[ G^-(n\lambda) = S_{\lambda}s(\emptyset). \] (37)

Hence we see that the vectors \( G^-(n\lambda) \) form a canonical basis of the space of highest weight vectors of \( F_q \), which reduces at \( q = 1 \) to the basis of plethysms \( p_n[s_{\lambda}]. \) Therefore, the polynomial \( e_{n\lambda,\mu}(-q^{-1}) \) is a \( q \)-analogue of the scalar product \( \langle p_n[s_{\lambda}], s_{\mu} \rangle. \)

Now, by a theorem of Littlewood [Li], this scalar product is 0 if the \( n \)-core \( \mu_{(n)} \) is not empty, and if \( \mu_{(n)} = \emptyset \) then

\[ \langle p_n[s_{\lambda}], s_{\mu} \rangle = e_{n}(\mu) \langle s_{\lambda}, s_{\mu^{(0)}}, \cdots, s_{\mu^{(n-1)}} \rangle \] (38)

where \( (\mu^{(0)}, \ldots, \mu^{(n-1)}) \) is the \( n \)-quotient of \( \mu \) and \( e_{n}(\mu) \) is its \( n \)-sign. It follows that \( e_{n\lambda,\mu}(q) \) is a \( q \)-analogue of the Littlewood-Richardson coefficient

\[ c_{\mu^{(0)},\ldots,\mu^{(n-1)}}^{\lambda} := \langle s_{\lambda}, s_{\mu^{(0)}}, \cdots, s_{\mu^{(n-1)}} \rangle. \]

We shall therefore write

\[ c_{\mu^{(0)},\ldots,\mu^{(n-1)}}^{\lambda}(q) := e_{n\lambda,\mu}(q). \]

9.2 An \( n \)-ribbon tableau \( T \) of shape \( \lambda \) and weight \( \mu = (\mu_1, \ldots, \mu_r) \) is defined as a chain of partitions

\[ \emptyset = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda \]

such that \( \alpha^i / \alpha^{i-1} \) is a horizontal \( n \)-ribbon strip of weight \( \mu_i \). Graphically, \( T \) may be described by numbering each \( n \)-ribbon of \( \alpha^i / \alpha^{i-1} \) with the number \( i \) (see Figure 7). We denote by \( \text{Tab}_n(\lambda, \mu) \) the set of \( n \)-ribbon tableaux of shape \( \lambda \) and weight \( \mu \). The spin of a ribbon tableau \( T \) is defined as the sum of the spins of its ribbons. Let

\[ L_{\lambda,\mu}(q) := \sum_{T \in \text{Tab}_n(\lambda, \mu)} q^{\text{spin}(T)} \] (39)

be the polynomial obtained by enumerating \( n \)-ribbon tableaux of shape \( \lambda \) and weight \( \mu \) according to spin. It follows immediately from (18) and (19) that

\[ V_{\mu} s(\emptyset) = \sum_{\lambda} L_{\lambda,\mu}(-q^{-1}) s(\lambda). \] (40)

By (36) and (37) we obtain

\[ e_{n\lambda,\mu}(q) = c_{\mu^{(0)},\ldots,\mu^{(n-1)}}^{\lambda}(q) = \sum_{\nu} \kappa_{\lambda,\nu} L_{\mu,\nu}(q). \] (41)
Example 19 Let \( n = 3, \) and \( \mu = (3^3, 2, 1) \), so that \( (\mu^{(0)}, \mu^{(1)}, \mu^{(2)}) = ((1), (1^2), (1)) \). The polynomials \( L_{(\lambda^3, 2, 1), \nu}(q) \) are as follows (see Figure 8)

\[
L_{(\lambda^3, 2, 1), (4)}(q) = 0, \\
L_{(\lambda^3, 2, 1), (3, 1)}(q) = q^7, \\
L_{(\lambda^3, 2, 1), (2^2)}(q) = q^7 + q^5, \\
L_{(\lambda^3, 2, 1), (2, 1^2)}(q) = 2q^7 + 2q^5 + q^3, \\
L_{(\lambda^3, 2, 1), (1^4)}(q) = 3q^7 + 5q^5 + 3q^3 + q.
\]

It follows that

\[
e_{(9,3), (\lambda^3, 2, 1)}(q) = \sum_{(1), (1^2), (1)} c_{(1), (1^2), (1)}^1 L_{(\lambda^3, 2, 1), (3, 1)}(q) - L_{(\lambda^3, 2, 1), (4)}(q) = q^7, \\
e_{(6^2), (\lambda^3, 2, 1)}(q) = \sum_{(1), (1^2), (1)} c_{(1), (1^2), (1)}^2 L_{(\lambda^3, 2, 1), (2^2)}(q) - L_{(\lambda^3, 2, 1), (3, 1)}(q) = q^5, \\
e_{(6,3^2), (\lambda^3, 2, 1)}(q) = \sum_{(1), (1^2), (1)} c_{(1), (1^2), (1)}^{(1^2)} L_{(\lambda^3, 2, 1), (2^1^2)}(q) - L_{(\lambda^3, 2, 1), (2^2)}(q) - L_{(\lambda^3, 2, 1), (3, 1)}(q) + L_{(\lambda^3, 2, 1), (4)}(q) = q^5 + q^3, \\
e_{(3^4), (\lambda^3, 2, 1)}(q) = \sum_{(1), (1^2), (1)} c_{(1), (1^2), (1)}^{(1)} L_{(\lambda^3, 2, 1), (1^4)}(q) - 3 L_{(\lambda^3, 2, 1), (2^1^2)}(q) + 2 L_{(\lambda^3, 2, 1), (2^2)}(q) + 3 L_{(\lambda^3, 2, 1), (3, 1)}(q) - L_{(\lambda^3, 2, 1), (4)}(q) = q.
\]

However in the case \( n = 2 \), the combinatorics of domino tableaux being easier than that of general \( n \)-ribbon tableaux, it is possible to reduce (41) to a positive sum. In [CL] a special class of domino tableaux called Yamanouchi domino tableaux was introduced. The set of Yamanouchi domino tableaux of weight \( \lambda \) and shape \( \mu \), denoted by \( \text{Yam}_2(\mu, \lambda) \), is in one to one correspondence with the set of Littlewood-Richardson tableaux counting the multiplicity of \( s_\lambda \) in \( s_\mu^{(0)} s_\mu^{(1)} \), but this bijection is not straightforward (see [vL]). In terms of these, one has

\[
e_{2\lambda, \mu}(q) = c_{\mu^{(0)}, \mu^{(1)}}^\lambda(q) = \sum_{Y \in \text{Yam}_2(\mu, \lambda)} q^{\text{spin}(Y)}. \tag{42}
\]
Note also that the equality
\[ K_{\lambda,\mu}(q^2) = \sum_\nu \kappa_{\lambda,\mu,\nu} L_{\eta_{\nu}}(q), \quad (n \geq \ell(\mu)) \]  
expressing the Kostka-Foulkes polynomial in terms of ribbon tableaux was first proved in [LLT2] using a different method.

## 10 Formulas for $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ in some good weight spaces

In Section 9 we have calculated $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ for some partitions $\lambda$ and $\mu$ with empty $n$-core. Quite opposite to this case, one can also compute explicitly $d_{\lambda,\mu}(q)$ and $e_{\lambda,\mu}(q)$ for partitions $\lambda$ and $\mu$ with certain 'large $n$-cores'. The expression is in terms of the classical Littlewood-Richardson coefficients.

For $k \geq 2$, let $\gamma(k)$ be the partition containing the following parts:

- the first $k - 1$ integers with multiplicity $n - 1$,
- the next $k - 1$ integers going 2 by 2 with multiplicity $n - 2$,
- the next $k - 1$ integers going 3 by 3 with multiplicity $n - 3$,
- \ldots
- the next $k - 1$ integers going $n - 1$ by $n - 1$ with multiplicity 1.

For example, if $n = 5$ and $k = 4$,
\[ \gamma(4) = (1^4, 2^4, 3^3, 4^3, 7^2, 12^2, 15^2, 18^2, 22, 26, 30). \]

For $n = 2$, $c(k) = (1, 2, \ldots, k - 1)$ is a staircase partition, that is, a 2-core partition. It is easy to check that more generally, for any $n$, $\gamma(k)$ is an $n$-core partition.

**Theorem 20 (LM)** Let $\lambda$ and $\mu$ be partitions such that
\[ \lambda(n) = \mu(n) = \gamma(k) \quad \text{and} \quad \sum_{i=0}^{n-1} |\lambda^{(i)}| = \sum_{i=0}^{n-1} |\mu^{(i)}| \leq k. \]  

Then
\[ d_{\lambda,\mu}(q) = q^{\delta(\lambda,\mu)} \sum_{\alpha^0, \ldots, \alpha^n} \prod_{0 \leq j \leq n-1} ^{\alpha^{(j)}} \beta^{(j)} \end{equation}  
where $\alpha^0, \ldots, \alpha^n, \beta^0, \ldots, \beta^{n-1}$ run through $\mathcal{P}$ subject to the conditions
\[ |\alpha^i| = \sum_{0 \leq j \leq i-1} |\lambda^{(j)}| - |\mu^{(j)}|, \quad |\beta^i| = |\mu^{(i)}| + \sum_{0 \leq j \leq i-1} |\mu^{(j)}| - |\lambda^{(j)}|, \]
and
\[ \delta(\lambda,\mu) := \sum_{0 \leq j \leq n-2} (n - 1 - j)(|\lambda^{(j)}| - |\mu^{(j)}|). \]

We also have
\[ e_{\lambda,\mu}(q) = q^{\Delta(\lambda,\mu)} \sum_{\alpha} \prod_{0 \leq k \leq n-1} ^{\alpha_k^{(k)}} \beta_{\lambda^{(k)}}, \]  

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where the sum is over all families of partitions \( \alpha = (\alpha_i^j \in \mathcal{P}, 0 \leq i \leq j \leq n - 1) \) subject to the conditions

\[
\sum_j k \alpha_i^j = |\mu^{(i)}|, \quad \sum_i |\alpha_i^j| = |\lambda^{(j)}|,
\]

and

\[
\Delta(\lambda, \mu) = \sum_{0 \leq j \leq n-1} j(|\lambda^{(j)}| - |\mu^{(j)}|).
\]

**Example 21** Let \( n = 2 \) and \( k = 3 \). We have \( \gamma(3) = (2, 1) \). The list of partitions with 2-core \((2, 1)\) and with weight \(|\gamma(3)| + 3 \cdot 2 = 9\) is

\((8, 1), (6, 3), (6, 1^3), (4, 3, 2), (4, 3, 1^2), (4, 2^2, 1), (4, 1^5), (3^2, 2, 1), (2^3, 1^3), (2, 1^7),\)

and the corresponding \( d_{\lambda, \mu}(q) \) are

\[
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11 Comparison with the Macdonald polynomials

11.1 As mentioned in Section 3, the classical Fock space $\mathcal{F}$ can be identified in a natural way to the ring of symmetric functions. Section 9 suggests yet another way of seeing symmetric functions as vectors of $\mathcal{F}_q$, namely by mapping the Schur function $s_\lambda$ to the canonical base vector $G^- (n \lambda)$. This provides a linear embedding $\iota$ of $\text{Sym}$ in $\mathcal{F}_q$ such that $\iota (s_\lambda)$ is the space of highest weight vectors under the action of $U_q$. We have $\iota (h_{\lambda}) = V_{\lambda} s (\emptyset)$, $\iota (s_\lambda) = S_{\lambda} s (\emptyset)$ and $\iota (p_{\lambda}) = B^-_{\lambda} s (\emptyset)$, where

$$B^-_{\lambda} := B_{-\lambda_1} \ldots B_{-\lambda_r},$$

and $B_k (k \in \mathbb{Z}^+)$ is the $q$-boson operator of $[\text{KMS}]$.

The natural scalar product on $\mathcal{F}_q$ is given by

$$\langle s (\lambda), s (\mu) \rangle = \delta_{\lambda, \mu}.$$

This is the scalar product with respect to which the canonical bases $B$ and $B^-$ are ‘almost orthonormal’ in the sense of $[\text{Lu3}]$, 14.2.1, that is, $B$ is orthonormal at $q = 0$ and $B^-$ is orthonormal at $q = \infty$.

**Proposition 23** We have

$$\langle \iota (p_{\lambda}), \iota (p_{\mu}) \rangle = \delta_{\lambda, \mu} z_{\lambda} \prod_{i=1}^{\ell (\lambda)} \frac{1 - q^{-2n_i \lambda_i}}{1 - q^{-2n_i \mu_i}},$$

where $z_{\lambda} = \prod_i q^{m_i (\lambda)} m_i (\lambda)!$.

**Proof** — It is proved in $[\text{KMS}]$ that the $q$-bosons operators satisfy the commutation rule

$$[B_k, B_l] = \delta_{k, -l} k \frac{1 - q^{-2n|k|}}{1 - q^{-2|k|}}.$$

It is also known that $B_k$ and $B_{-k}$ are adjoint to each other with respect to $\langle \cdot, \cdot \rangle$. It follows that

$$\langle \iota (p_{\lambda}), \iota (p_{\mu}) \rangle = \langle B^-_{\lambda} s (\emptyset), B^-_{\mu} s (\emptyset) \rangle = \langle s (\emptyset), B^+_{\lambda} B^-_{\mu} s (\emptyset) \rangle,$$

where $B^+_{\lambda} := B_{\lambda_1} \ldots B_{\lambda_r}$. Now, since $B_k s (\emptyset) = 0$ for $k > 0$, we have

$$B_k B_{-k} s (\emptyset) = [B_k, B_{-k}] s (\emptyset) = k \frac{1 - q^{-2nk}}{1 - q^{-2k}} s (\emptyset).$$

It follows by induction on $r$ that

$$B_k B^r_{-k} s (\emptyset) = r k \frac{1 - q^{-2nrk}}{1 - q^{-2rk}} B^{r-1}_{-k} s (\emptyset),$$

and therefore

$$B^r_k B^r_{-k} s (\emptyset) = r! k^r \left( \frac{1 - q^{-2nk}}{1 - q^{-2k}} \right)^r s (\emptyset).$$

The result follows because $B_k$ and $B_{-l}$ commute when $k \neq l$. \hfill $\Box$
11.2 Put \( t = q^{-2} \) and \( p = q^{-2n} = t^n \). Then by Proposition 23, for any symmetric functions \( f \) and \( g \) we have

\[
\langle \iota(f), \iota(g) \rangle = \langle f, g \rangle_{p,t}
\]

where \( \langle \cdot, \cdot \rangle_{p,t} \) is Macdonald’s scalar product ([Med], p. 306). In other words Macdonald’s scalar product for this choice of parameters can be seen as the restriction of the natural scalar product of the Fock space representation of \( U_q(\hat{sl}_n) \) to its subspace of highest weight vectors. This implies, using (37) and (41), that the Macdonald scalar product of two Schur functions can be expressed in terms of parabolic Kazhdan-Lusztig polynomials, or \( q \)-Littlewood-Richardson coefficients:

**Corollary 24**

\[
\langle s_\lambda, s_\mu \rangle_{p,t} = \sum_\nu e_{n\lambda,\nu}(q^{-1}) e_{n\mu,\nu}(q^{-1}) = \sum_\nu c^\lambda_{\nu(0),\ldots,\nu(n-1)}(q^{-1}) c^\mu_{\nu(0),\ldots,\nu(n-1)}(q^{-1}).
\]

Let \( A^- \) be the subring of \( \mathbb{K} \) consisting of the rational functions without pole at \( v = \infty \), and denote by \( \mathcal{L}^- \) the free \( A^- \)-module with basis \( \{s(\lambda)\} \). Let \( P_\lambda = P_\lambda(p,t) \) denote the Macdonald symmetric function corresponding to our choice of parameters.

**Proposition 25** The images \( \iota(P_\lambda) \) of the Macdonald symmetric functions satisfy the two conditions

\[
\overline{\iota(P_\lambda)} = \iota(P_\lambda), \quad \iota(P_\lambda) \equiv s(n\lambda) \mod q^{-1}\mathcal{L}^-.
\]

**Proof** — The basis \( \{P_\lambda\} \) can be obtained via the Gram-Schmidt orthogonalization process applied to the basis \( \{m_\lambda\} \), or equivalently to the basis \( \{s_\lambda\} \). This gives

\[
P_\mu = s_\mu + \sum_{\lambda < \mu} \chi_{\lambda,\mu} s_\lambda
\]

where

\[
\chi_{\lambda,\mu} = -D_{\lambda,\mu}/D_\mu, \quad D_\mu = \det \left[ \langle s_\alpha, s_\beta \rangle_{p,t} \right]_{\alpha,\beta < \mu}
\]

and \( D_{\lambda,\mu} \) is obtained from \( D_\mu \) by replacing \( \beta \) by \( \mu \) in the column \( \beta = \lambda \). Let \( R \) be the subfield of \( \mathbb{K} \) consisting of the rational functions invariant under \( q \mapsto q^{-1} \). We have \( \langle p_\lambda, p_\mu \rangle_{p,t} \in q^{-(n-1)k}\mathbb{K}[R] \). Since \( s_\lambda \) is a \( \mathbb{Q} \)-linear combination of \( p_\mu \)'s, we also have \( \langle s_\lambda, s_\mu \rangle_{p,t} \in q^{-(n-1)k}\mathbb{K}[R] \) for any partitions \( \lambda \) and \( \mu \) of \( k \). Hence \( \chi_{\lambda,\mu} \in R \), and since \( \iota(s_\lambda) = G^-(n\lambda) \) is bar invariant, we get that \( \iota(P_\lambda) \) is bar invariant.

Finally, since \( B^- \) is almost orthonormal at \( q = \infty \), we have

\[
\langle s_\lambda, s_\mu \rangle_{p,t} \equiv \delta_{\lambda,\mu} \mod q^{-1}\mathbb{K}[q^{-1}].
\]

Therefore \( D_\mu \equiv 1, D_{\lambda,\mu} \equiv 0 \mod q^{-1}\mathbb{K}[q^{-1}] \), hence \( \iota(P_\lambda) \equiv s(n\lambda) \mod q^{-1}A^- \) as required. \( \square \)

Note that \( \iota(P_\lambda) \) satisfies conditions very similar to the defining properties of \( G^-(n\lambda) \). The only difference is that \( G^-(n\lambda) \) is required to belong to \( \mathcal{L}^- \) while \( \iota(P_\lambda) \) is allowed to belong to the larger lattice \( \mathcal{L}^- \). On the other hand, the vectors \( \iota(P_\lambda) \) are pairwise orthogonal with respect to \( \langle \cdot, \cdot \rangle \) whereas the vectors \( G^-(n\lambda) \) are only orthogonal at \( q = \infty \).

The results of this section are very similar to the main results of [BFJ], but not obviously the same. Here we have used a Heisenberg algebra of intertwining operators of the \( U_q \)-module \( \mathcal{F}_q \), whereas [BFJ] uses a Heisenberg subalgebra of \( U_q \) coming from Drinfeld’s new realization. Also,
we work in the so-called ‘principal picture’ while [BFJ] considers the ‘homogeneous picture’. Finally, we obtain Macdonald’s scalar product \( \langle \cdot, \cdot \rangle_{U_2} \) by working with \( \hat{S}_k \) at level 1, while in [BFJ] the scalar product \( \langle \cdot, \cdot \rangle_{U_2} \) is expected to be associated with \( \hat{S}_k \) at level \( k \) (in [BFJ], only the case \( k = 1 \) is treated). This is probably another instance of the classical ‘level-rank duality’ due to Frenkel [F1, F2]. The higher level \( q \)-deformed Fock spaces of Takemura and Uglov [U, TU] provide the natural setting for the quantized version of the level-rank duality, and we should probably obtain some deeper understanding by trying to interpret the Macdonald polynomials in this broader context.

References


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