Theory of Computation — Michael Sipser
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Sample MidTerm Examination Questions

1. (a) Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and let $A = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0, \text{ and } i = j \text{ or } i = k\}$.
   Describe (in English) a pushdown automaton that recognizes A.

   (b) Let R be the regular expression $\Sigma^*\mathtt{1100}\Sigma^*$ where $\Sigma = \{\mathtt{0}, \mathtt{1}\}$.
   Let $D = L(R)$ and let $E = \overline{D}$, the complement of D.
   Give the state diagram of a DFA with at most 5 states that recognizes E.

2. Let $\Sigma = \{\mathtt{(}, \mathtt{)}\}$ and let P be the language consisting of all strings of properly nested parentheses. For example, P contains "()()", "((()))", "(()((()(()))))" and "ε", but not ")(" and "((((".

   (a) Give a CFG that generates P.

   (b) Show that P is not a regular language.

3. (a) Let $A = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^i \mid i \leq j \leq 2i\}$.
   Prove that A is not a context-free language.

   (b) Let $B = \{\mathtt{a}^i\mathtt{b}^j \mid i \leq j \leq 2i\}$.
   Give an *unambiguous* context-free grammar generating B.

4. Let $D = \{\langle M\rangle \mid M \text{ is a TM that accepts the input string } \mathtt{101}\}$.

   (a) Show that D is undecidable.
   (Do not use Rice's theorem. If you don't know Rice's theorem, ignore this comment.)

   (b) Show that the complement of D is not Turing-recognizable.

5. A *2-way pushdown automaton* (2WAY-PDA) is a nondeterministic pushdown automaton that has a single stack and that can move its input head in both directions on the input tape. In addition we assume that a 2WAY-PDA is capable of detecting when its input head is at either end of its input tape. A 2WAY-PDA accepts its input by entering an accept state.

   (a) Show that a 2WAY-PDA can recognize the language $\{\mathtt{a}^m\mathtt{b}^m\mathtt{c}^m \mid m \geq 0\}$.

   (b) Let $E_{\text{2WAY-PDA}} = \{\langle P\rangle \mid P \text{ is a 2WAY-PDA which recognizes the empty language}\}$.
   Show that $E_{\text{2WAY-PDA}}$ is not decidable.

6. Consider the infinite two-dimensional grid, $G = \{(m, n) \mid m \text{ and } n \text{ are integers}\}$. Every point in G has 4 neighbors, North, South, East, and West, obtained by varying m or n by ± 1. Starting at the origin $(0, 0)$, a string of commands N, S, E, W, generates a path in G. For example, the string NESW, generates a path clockwise around a unit square touching the origin. Say that a path is *closed* if it starts at the origin and ends at the origin.

   Let C be the collection of all strings over $\Sigma = \{\mathtt{N}, \mathtt{S}, \mathtt{E}, \mathtt{W}\}$ that generate a closed path.

   (a) Give a clear mathematical description of C as a language.

   (b) Describe in English two CFLs, A and B, such that $C = A \cap B$.
   Give a CFG that generates A.

   (c) Prove that C is not context-free.

7. Let $\Sigma = \{\mathtt{0}, \mathtt{1}\}$. Consider the problem of testing whether a PDA accepts some string of the form $\{w \mid w \in \mathtt{0}^*\mathtt{1}^*\}$. Is this problem decidable? Prove your answer.