Reminder

• In class we introduced non-deterministic finite automata (NFA). NFAs are allowed to have multiple (or none) transitions for the same symbol from the same state, and are also allowed to have "free" $\epsilon$ transitions. We showed that NFAs are equivalent to DFAs in terms of the languages they can recognize - the regular languages.

• Closure Properties of regular languages: If $A, B$ are regular, then so are their union ($A \cup B$), concatenation ($A \circ B = AB$), star ($A^*$), intersection ($A \cap B$), complement ($\overline{A}$), reversal ($A^R$), and difference ($A \setminus B$). We will show the last three in these notes.

• The pumping lemma may be used to show non-regularity of languages.

Lemma 1 (The Pumping Lemma). If $A$ is a regular language, then there is a number $p$ (pumping length) such that if $s \in A$ and $|s| \geq p$, then $s$ can be written as $s = xyz$ with the following three properties:

1. $xy^iz \in A$, for $i \geq 0$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

• Examples of some Non-Regular languages:
  - $\{1^n0^n \mid n \geq 0\}$, and
  - $\{w \mid w$ has $=$ numbers of 0s and 1s$\}$.

Example 1 — The Regular Languages are closed under complement

If $A$ is any language, let $\overline{A}$ be the complement of $A$. Formally, $\overline{A} = \{w \mid w \notin A\}$.

Prove that the regular languages are closed under complement. Namely if $A$ is regular, then $\overline{A}$ is also regular.

Solution

Given a DFA $M$ that recognizes $A$, we construct another DFA $N$ that recognizes $\overline{A}$. $N$ is the same as $M$, except that its accepting and non-accepting states are switched. Formally, if $M = (Q, \Sigma, \delta, q_0, F)$, then we have $N = (Q, \Sigma, \delta, q_0, Q \setminus F)$. 
Example 2 — The Regular Languages are closed under reversal

If $A$ is any language, let $A^R$ be the reversal language of $A$. Formally, $A^R = \{w \mid w^R \in A\}$, where for $w = w_1w_2 \cdots w_n$, $w^R = w_nw_{n-1} \cdots w_1$.

Prove that the regular languages are closed under reversal. Namely, if $A$ is regular, then $A^R$ is also regular.

Solution

Given a DFA $M$ that recognizes $A$, we describe a NFA $N$ that recognizes $A^R$ using the following idea. The transition function will be the “inverse” of the original transition function. We turn the start state into the accepting state and the accepting states into the start state. But what if $M$ had more than one accepting state? We add additional state to be the new single starting state, and connect it via $\varepsilon$-transitions to the accepting states of $M$. The formal definition follows.

Let $M = (Q, \Sigma, \delta, q_0, F)$. Construct NFA $N = (Q \cup \{t\}, \Sigma, \delta', t, F')$, for $t \notin Q$, that recognizing the reversal of $A$ as follows:

1. $F' = \{q_0\}$.
2. $\delta'(q, a) = \{r \in Q \mid \delta(r, a) = q\}$ if $q \neq t$, and $\delta'(t, \varepsilon) = F$.

Example 3 — The Regular Languages are closed under difference

If $A$ and $B$ are languages, let $A\setminus B$ be the difference between $A$ and $B$. Formally, $A\setminus B = \{w \mid w \in A \text{ and } w \notin B\}$.

Prove that the regular languages are closed under difference. Namely, if $A$ and $B$ are regular, then $A\setminus B$ is also regular.

Solution

We can easily show this by using other closure properties. Namely, note that $A\setminus B = A \cap \overline{B}$. Since the regular languages are closed under intersection and complement, it follows that they are also closed under difference.

Example 4 — Some Non-Regular Languages

Prove that the following languages are non-regular.

1. $L_1 = \{0^n1^m0^n \mid n, m \geq 0\}$.
2. $L_2 = \{0^i1^j \mid i \geq j \geq 0\}$.
3. $L_3 = \{w \mid w \text{ has } \neq \text{ numbers of } 0s \text{ and } 1s\}$.
4. $L_4 = \{a^mb^n \mid m \neq n\}$
Solutions

1. We use the pumping lemma to “pump up”. Assume $L_1$ is regular. Use the pumping lemma to get a pumping length $p$ satisfying the conditions of the pumping lemma. Set $s = 0^p1^p0^p$. Obviously, $s \in L_1$ and $|s| \geq p$. Thus, the pumping lemma implies that the string $s$ can be written as some concatenation of strings $xyz$. From condition 3 of the pumping lemma, we have $x = 0^a$, $y = 0^b$ and $z = 0^c1^p$, where $b \geq 1$ and $a + b + c = p$. However, the string $s' = xy^2z \notin L_1$, since $a + 2b + c > p$, and thus we do not have an equal number of 0s on both ends of the string. That contradicts the pumping lemma. Thus $L_1$ is not regular.

2. We use the pumping lemma to “pump down”. Assume $L_2$ is regular. Use the pumping lemma to get a pumping length $p$ satisfying the conditions of the pumping lemma. Set $s = 0^p1^p$. Obviously, $s \in L_2$ and $|s| \geq p$. Thus, the pumping lemma implies that the string $s$ can be written as some concatenation of strings $xyz$. From condition 3 of the pumping lemma, we have $x = 0^a$, $y = 0^b$ and $z = 0^c1^p$, where $b \geq 1$ and $a + b + c = p$. However, the string $s' = xy^0z = 0^a+c1^p \notin L_2$, since $a + c < p$. That contradicts the pumping lemma. Thus $L_2$ is not regular.

3. We use the closure properties of regular languages. Assume $L_3$ is regular. Since the regular languages are closed under complement, $\overline{L_3} = \{w \mid w \text{ has } = \text{numbers of 0s and 1s}\}$ is regular. But, we saw in class that the latter language is not regular, a contradiction. Thus $L_3$ is not regular.

4. We use the closure properties of regular languages. Assume $L_4$ is regular. Since the regular languages are closed under difference, $a^*b^* \setminus L_4 = a^n b^m$ must be regular, since $a^*b^*$ is also regular. However, we know that the resulting language is not regular, a contradiction. Thus, $L_4$ cannot be regular.

Other

In my recitation, the question of whether or not non regular languages are closed under the same operations as the regular languages are came up. The answer is not necessarily. For instance, the non regular languages are not closed under the regular operations (union, concatenation, and star) as well as some other operations (eg. intersection), but are closed under others (eg. complement and reversal). We will explain why for union, complement, and reversal.

To see why non regular languages are not closed under union, we show an example. Consider a non regular language $A$ and its complement $\overline{A}$. These are two non regular languages. However, $A \cup \overline{A} = \Sigma^*$, which is regular. Thus, the non regular languages are not closed under the union operation.

To see why non regular languages are closed under complement and reversal, consider what happens if you assume that they aren’t, and then try to apply the operation twice. You should be able to easily see a contradiction.