1. Recall the language $D = \{ \langle p \rangle | p \text{ is a polynomial in several variables having an integral root} \}$. We stated, but didn’t prove, that $D$ is undecidable. In this problem, you are to prove a different property of $D$—namely, that $D$ is $\text{NP-hard}$. A problem is $\text{NP-hard}$ if all problems in NP are polynomial time reducible to it, even though it may not be in NP itself. So you must show that all problems in NP are polynomial time reducible to $D$.

2. For a cnf-formula $\phi$ with $m$ variables and $c$ clauses, show that you can construct in polynomial time an NFA with $O(cm)$ states that accepts all nonsatisfying assignments, represented as Boolean strings of length $m$. Conclude that $P \neq \text{NP}$ implies that NFAs cannot be minimized in polynomial time. Here, minimizing an NFA means finding an NFA with the fewest possible number of states that recognizes the same language as a given NFA.

3. The cat-and-mouse game is played by two players, “Cat” and “Mouse,” on an undirected graph. At a given point, each player occupies a node of the graph. The players take turns moving to a node adjacent to the one that they currently occupy. Cat wins if the two players ever occupy the same node. Mouse wins if a situation repeats (i.e., the two players simultaneously occupy positions that they simultaneously occupied previously, and it is the same player’s turn to move). $\text{HAPPY-CAT} = \{ \langle G, c, m \rangle \mid G, c, m \text{ are respectively a graph, and positions of the Cat and Mouse, such that Cat has a winning strategy if Cat moves first} \}$. Show that $\text{HAPPY-CAT}$ is in P. (Hint: The solution is not complicated and doesn’t depend on subtle details in the way the game is defined. Consider the entire game tree. It is exponentially big, but you can search it in polynomial time.)

4. The Japanese game go-moku is played by two players, $\text{X}$ and $\text{O}$, on a $19 \times 19$ grid. Players take turns placing markers, and the first player to achieve five of her markers consecutively in a row, column, or diagonal is the winner. Consider this game generalized to an $n \times n$ board. Let $\text{GM} = \{ \langle B \rangle \mid B \text{ is a configuration in generalized go-moku, where } \text{X} \text{ has a winning strategy} \}$. By a configuration we mean a board with markers placed on it, such as may occur in the middle of a play of the game, together with an indication of which player moves next. Show that $\text{GM} \in \text{PSPACE}$.

5. Let $\text{ALL}_{\text{DFA}} = \{ \langle M \rangle \mid M \text{ is a DFA and } L(M) = \Sigma^* \}$.

Show that $\text{ALL}_{\text{DFA}}$ is NL-complete.

6. (a) Let $\text{ADD} = \{ \langle x, y, z \rangle \mid x, y, z > 0 \text{ are binary integers and } x + y = z \}$. Show $\text{ADD} \in \text{L}$.

(b) Let $\text{PAL-ADD} = \{ \langle x, y \rangle \mid x, y > 0 \text{ are binary integers where } x + y \text{ is an integer whose binary representation is a palindrome} \}$. (Note that the binary representation of the sum is assumed not to have leading zeros. A palindrome is a string that equals its reverse.) Show that $\text{PAL-ADD} \in \text{L}$.

7. (optional) Give an example of an NL-complete context free language.

(Hint: Read the solution to Problem 8.34 regarding the $\text{CYCLE}$ language.)