Problem Set 3

1. Consider the problem of determining whether a single-tape Turing machine ever writes a blank symbol over a nonblank symbol during the course of its computation on any input string. Formulate this problem as a language and show that it is undecidable.

2. Let $AMBIG_{\text{CFG}} = \{\langle G \rangle | G$ is an ambiguous CFG$\}$. Show that $AMBIG_{\text{CFG}}$ is undecidable. (Hint: Use a reduction from $PCP$. Given an instance $P = \{[t_1, b_1], [t_2, b_2], \ldots, [t_k, b_k]\}$ of the Post Correspondence Problem, construct a CFG $G$ with the rules

$$S \rightarrow T \mid B$$
$$T \rightarrow t_1Ta_1 \mid \cdots \mid t_kTa_k \mid t_1a_1 \mid \cdots \mid t_ka_k$$
$$B \rightarrow b_1Ba_1 \mid \cdots \mid b_ka_k$$

where $a_1, \ldots, a_k$ are new terminal symbols. Prove that this reduction works.)

3. Let $A$ be a language.

   (a) Show that $A$ is Turing-recognizable iff $A \leq_m A_{\text{TM}}$.
   (b) Show that $A$ is decidable iff $A \leq_m 0^*1^*$.

4. (a) Let $J = \{w | \text{either } w = 0x \text{ for some } x \in A_{\text{TM}}, \text{ or } w = 1y \text{ for some } y \in \overline{A_{\text{TM}}}$\}. Use mapping reductions to show that neither $J$ nor $\overline{J}$ is Turing-recognizable.
   (b) Let $FINITE_{\text{TM}} = \{\langle T \rangle | T \text{ is a TM and } L(T) \text{ is a finite language}$\}. Show that $A_{\text{TM}} \leq_m FINITE_{\text{TM}}$ to prove that $FINITE_{\text{TM}}$ is not T-recognizable.
   (c) (harder) Show that $\overline{FINITE_{\text{TM}}}$ is not T-recognizable.

5. An erasing Turing machine ($e\text{TM}$) operates like an ordinary deterministic TM except that when its head reads a symbol on the tape, the $e\text{TM}$ may either leave that symbol unchanged or it may change it to the usual blank symbol ($\#$). Changing a tape symbol to a nonblank symbol is prohibited. Formally speaking, the $e\text{TM}$ transition function $\delta$ may have $\delta(q, a) = (r, b, M)$ only where $b = a$ or $b = \#$ for $q, r \in Q$ and $M \in \{L, R\}$.

   (a) Let $B = \{\#a^k\#^k \mid k \geq 0\}$ and $\Sigma = \{a, b, \#\}$. Briefly describe an $e\text{TM}$ that decides $B$.
   (b) Let $E_{\text{TM}} = \{\langle B \rangle | B \text{ is an } e\text{TM where } L(B) = \emptyset\}$. Prove that $E_{\text{TM}}$ is undecidable.

   (Provide enough detail to show how your proof depends on the $e\text{TM}$ model.)

6. Give an example in the spirit of the recursion theorem of a program in a real programming language (or a reasonable approximation of a programming language) that prints itself out.

7* (optional) Define an $e\text{TM}$ as in #5. Let $A_{e\text{TM}} = \{\langle B, w \rangle | B \text{ is an } e\text{TM that accepts input } w\}$. Prove that $A_{e\text{TM}}$ is decidable. Your solution must handle the possibility that the $e\text{TM}$ moves its head to the right of the input, into the originally blank part of the tape.

Midterm exam: Thursday, October 25, 2018, 2:30–4:00, top floor of Walker.
Covers Chapters 1, 2 (except 2.4), 3, 4, 5, and 6.1.

Final exam: Monday, December 17, 2018, 1:30–4:30, duPont Gym.
Covers the above plus Chapters 7, 8, 9.1, 9.2, 10.2 (except the section on primality), and 10.4 through Theorem 10.33.