Read all of Chapter 5 and Section 6.1.

0.1 Read and solve, but do not turn in: Book, 5.14.

1. If $w$ is a string over an alphabet $\Sigma$, let \( \text{middle}(w) = a \) where $a \in \Sigma$ and $w = xay$ for $|x| = |y|$. If $A$ is any language, let \( \text{MIDDLE0}(A) = \{ w \mid w \in A \text{ and } \text{middle}(w) = 0 \} \).

   (a) If $A$ is a regular language, show that $\text{MIDDLE0}(A)$ is a CFL.
   (b) Consider the problem of testing whether for a given DFA $D$ there exists some $w \in L(D)$ where $\text{middle}(w) = 0$. Show that this problem is decidable. (Hint: Use part a.)

2. Let $\text{DISJOINT}_{\text{CFG}} = \{ (G, H) \mid G \text{ and } H \text{ are CFGs and } L(G) \cap L(H) = \emptyset \}$. Show that $\text{DISJOINT}_{\text{CFG}}$ is undecidable. (Hint: Use a reduction from $\text{PCP}$. Given an instance $P = \{ [\frac{t_1}{b_1}], [\frac{t_2}{b_2}], \ldots, [\frac{t_k}{b_k}] \}$ of the Post Correspondence Problem, construct CFGs $G$ and $H$ with the rules

\[
G: \quad T \rightarrow t_1Ta_1 | \cdots | t_kTa_k | t_1a_1 | \cdots | t_ka_k
\]

\[
H: \quad B \rightarrow b_1Ba_1 | \cdots | b_kBa_k | b_1a_1 | \cdots | b_ka_k
\]

where $a_1, \ldots, a_k$ are new terminal symbols. Prove that this reduction works.)

3. Let $A$ be a language.

   (a) Show that $A$ is Turing-recognizable iff $A \leq_m A_{\text{TM}}$.
   (b) Show that $A$ is decidable iff $A \leq_m 0^*1^*$.

4. Let $\text{DECIDER}_{\text{TM}} = \{ (M) \mid M \text{ is a TM that halts on every input} \}.

   (a) Show that $A_{\text{TM}} \leq_m \text{DECIDER}_{\text{TM}}$ to prove that $\text{DECIDER}_{\text{TM}}$ is not T-recognizable.
   (b) (harder) Show that $A_{\text{TM}} \leq_m \overline{\text{DECIDER}_{\text{TM}}}$ to prove that $\overline{\text{DECIDER}_{\text{TM}}}$ is not T-recognizable.

5. In a two-dimensional finite automaton (2DIM-DFA) the input is an $m \times n$ rectangle, for any $m, n \geq 2$. The squares along the boundary of the rectangle contain the symbol \# and the internal squares contain symbols over the input alphabet $\Sigma$. The transition function $\delta: Q \times (\Sigma \cup \{\#\}) \rightarrow Q \times \{L, R, U, D\}$ indicates the next state and the new head position (Left, Right, Up, Down). The machine accepts when it enters one of the designated accept states. It rejects if it tries to move off the input rectangle or if it never halts. Two such machines are equivalent if they accept the same rectangles.

   (a) Let $A_{2\text{DIM-DFA}} = \{ (B,r) \mid B \text{ is a 2DIM-DFA and } B \text{ accepts rectangle } r \}$.
   Show that $A_{2\text{DIM-DFA}}$ is decidable.

   (b) Let $\text{EQ}_{2\text{DIM-DFA}} = \{ (B,C) \mid B \text{ and } C \text{ are equivalent 2DIM-DFAs} \}$.
   Show that $\text{EQ}_{2\text{DIM-DFA}}$ is not decidable.

6. Give an example in the spirit of the recursion theorem of a program in a real programming language (or a reasonable approximation of a programming language) that prints itself out.

7* (optional) Show that no computable function $f: \Sigma^* \rightarrow \Sigma^*$ exists where if enumerator $E$ enumerates a language in string order, then $f(\langle E \rangle) = \langle D \rangle$ where $D$ is a TM that decides the same language. (See the note in the solution to p-set 2 problem 5.)

Midterm exam: Thursday, October 24, 2019, 2:30–4:00, top floor of Walker.
Covers Chapters 1, 2 (except 2.4), 3, 4, 5, and 6.1.

Final exam: Monday, December 17, 2019, 9am–noon, Johnson Track.
Covers the above plus Chapters 7, 8, 9.1, 9.2, 10.2 (except the section on primality), and 10.4 through Theorem 10.33.