Problem Set 1

Read all of Chapters 1 and 2 except Section 2.4.

0.1 Read and solve, but do not turn in: Book, 1.14 . [swapping NFA accept/non-accept states]

0.2 Read and solve, but do not turn in: Book, 1.31 . [closure under reversal]

0.3 Read and solve, but do not turn in: Book, 1.46c . [Pumping lemma]

1. Let
\[ \Sigma_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 1 \\ 1 \\ \end{bmatrix} \right\}. \]

\( \Sigma_3 \) contains all size 3 columns of 0s and 1s. A string of symbols in \( \Sigma_3 \) gives three rows of 0s and 1s. Consider each row to be a binary number and let
\[ B = \{ w \in \Sigma_3^* | \text{the bottom row of } w \text{ is the sum of the top two rows} \} . \]

For example, \[ \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \end{bmatrix} \in B, \] but \[ \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \end{bmatrix} \notin B. \] Show that \( B \) is regular.

You may assume the result claimed in Problem 0.2 above.

2. Let \( \Sigma_3 \) be the same as in Problem 1. Let
\[ M = \{ w \in \Sigma_3^* | \text{the bottom row of } w \text{ is the product of the top two rows} \} . \]

For example, \[ \begin{bmatrix} 0 \\ 1 \\ 0 \\ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \end{bmatrix} \in M, \] but \[ \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \end{bmatrix} \notin M. \] Show that \( M \) is not regular.

3. Let \( \Sigma = \{ 0, 1 \} . \)
   (a) Let \( TUT = \{ tut | t, u \in \Sigma^* \} \). Show \( TUT \) is regular.
   (b) Let \( TUTU = \{ tutu | t, u \in \Sigma^* \} \). Show \( TUTU \) is not regular.

4. Let \( x \) and \( y \) be strings over some alphabet \( \Sigma \). Say \( x \) is a substring of \( y \) if \( y \in \Sigma^* x \Sigma^* \).
Define the avoids operation for languages \( A \) and \( B \) to be
\[ A \text{ avoids } B = \{ w | w \in A \text{ and } w \text{ doesn’t contain any string in } B \text{ as a substring} \} . \]

Prove that the class of regular languages is closed under the avoids operation.
(Hint: Theorems we’ve shown may be helpful. You may assume the results of 0.1 – 0.3 above.)

5. Is the class of non-regular languages closed under (a) union? (b) concatenation? (c) star? and (d) complementation? In each part, prove your answer.

6. Let \( M_1 \) and \( M_2 \) be DFAs that have \( k_1 \) and \( k_2 \) states, respectively, where \( L(M_1) \neq L(M_2) \).
Using an argument similar to the proof of the pumping lemma, show that there is a string \( s \) where \( |s| \leq k_1 k_2 \) and where exactly one of \( M_1 \) and \( M_2 \) accepts \( s \).

7. * (⋆ means optional) Improve the bound in Problem 6 to show that such an \( s \) exists where \( |s| \leq k_1 + k_2 \).