Problem Set 1

Read all of Chapters 1 and 2 except Section 2.4.

0.1 Read and solve, but do not turn in: Book, 1.14. [swapping NFA accept/non-accept states]

0.2 Read and solve, but do not turn in: Book, 1.31. [closure under reversal]

0.3 Read and solve, but do not turn in: Book, 1.46c. [Pumping lemma]

1. Let \( \Sigma_2 = \{ [0]_0, [0]_1, [1]_0, [1]_1 \} \). Here, \( \Sigma_2 \) contains all columns of 0s and 1s of height two. A string of symbols in \( \Sigma_2 \) gives two rows of 0s and 1s. Consider each row to be a binary number and let
\[
D = \{ w \in \Sigma_2^* \mid \text{the top row of } w \text{ is a larger number than is the bottom row} \}.
\]
For example, \([0] \quad [0] \quad [1] \quad [0] \in D\), but \([0] \quad [1] \quad [1] \quad [0] \notin D\). Show that \( D \) is regular.

2. Let \( \Sigma_2 \) be the same as in Problem 1. Let
\[
E = \{ w \in \Sigma_2^* \mid \text{the bottom row of } w \text{ is the reverse of the top row of } w \}.
\]
For example, \([1] \quad [1] \quad [0] \quad [1] \in E\), but \([1] \quad [1] \quad [1] \quad [0] \notin E\). Show that \( E \) is not regular.

3. Let \( A \) be any language. Define \( \text{DROP-ONE}(A) \) to be the language containing all strings that can be obtained by removing one symbol from a string in \( A \). Thus, \( \text{DROP-ONE}(A) = \{ xy \mid x, y \in \Sigma^* \} \).

Show that the class of regular languages is closed under the \( \text{DROP-ONE} \) operation.

Give both a proof by picture and a more formal proof by construction as in Theorem 1.47.

4. Let \( \Sigma = \{0, 1\} \). Let \( WW_k = \{ ww \mid w \in \Sigma^* \text{ and } w \text{ is of length } k \} \).

(a) Show that for each \( k \), no DFA can recognize \( WW_k \) with fewer than \( 2^k \) states.

(b) Describe a much smaller NFA for \( \overline{WW}_k \), the complement of \( WW_k \).

5. Let \( \Sigma = \{0, 1\} \).

(a) Let \( A = \{ 0^k u 0^k \mid k \geq 1 \text{ and } u \in \Sigma^* \} \). Show \( A \) is regular.

(b) Let \( B = \{ 0^k 1 u 0^k \mid k \geq 1 \text{ and } u \in \Sigma^* \} \). Show \( B \) is not regular.

6. Let \( \Sigma = \{0, 1\} \) and let \( C_1 = \{ trt \mid t \in 0^* \text{ and } r \in 0^* 10^* \}, \) where \( |t| = |r| \). The notation \( |x| \) means the length of string \( x \). Show that \( C_1 \) is a CFL, by giving a CFG and by giving a PDA. You do not need to prove that your solutions work, but please give comments to assist the grader. (Hint: This problem is tricky but not complicated. It has a CFG with three rules.)

7. \( \star \) (optional) For any language \( A \) let \( RC(A) = \{ xy \mid xy \in A \} \).

Show that the class of CFLs is closed under the \( RC \) operation.