Problem Set 1

Please turn in each problem on a separate page with your name.

Read all of Chapters 1 and 2 except Section 2.4.

0.1 Read and solve, but do not turn in: Book, 1.14. [swapping NFA accept/non-accept states]

0.2 Read and solve, but do not turn in: Book, 1.31. [closure under reversal]

1. Let \( \Sigma_3 = \{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \} \).

\( \Sigma_3 \) contains all size 3 columns of 0s and 1s. A string of symbols in \( \Sigma_3 \) gives three rows of 0s and 1s. Consider each row to be a binary number and let

\[ B = \{ w \in \Sigma_3^* \mid \text{the bottom row of } w \text{ is the sum of the top two rows} \} \]

For example, \( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in B \), but \( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin B \). Show that \( B \) is regular.

(Hint: Working with \( B^R \) is easier. You may assume the result claimed in Problem 1.31.)

2. Let \( \Sigma_3 \) be the same as in Problem 1. Let

\[ M = \{ w \in \Sigma_3^* \mid \text{the bottom row of } w \text{ is the product of the top two rows} \} \]

For example, \( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in M \), but \( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \notin M \). Show that \( M \) is not regular.

3. Let \( \Sigma = \{0, 1\} \).

(a) Let \( TUT = \{ tut \mid t, u \in \Sigma^* \} \). Show \( TUT \) is regular.

(b) Let \( TUTU = \{ tutu \mid t, u \in \Sigma^* \} \). Show \( TUTU \) is not regular.

4. For languages \( A \) and \( B \), let the **shuffle** of \( A \) and \( B \) be the language

\[ \{ w \mid w = a_1b_1 \cdots a_kb_k \text{ for } k \geq 0 \text{ where } a_1 \cdots a_k \in A \text{ and } b_1 \cdots b_k \in B, \text{ each } a_i, b_i \in \Sigma^* \} \]

Show that the class of regular languages is closed under shuffle.

5. Let \( \Sigma = \{0, 1\} \). Let \( WW_k = \{ wu \mid w \in \Sigma^* \text{ and } w \text{ is of length } k \} \).

Show that for each \( k \), no DFA can recognize \( WW_k \) with fewer than \( 2^k \) states.

Describe a much smaller NFA for \( \overline{WW}_k \), the complement of \( WW_k \).

6. (a) Use CFGs to show that the class of CFLs is closed under union.

(b) Let \( E = \{ a^i b^j \mid i \neq j \text{ and } 2i \neq j \} \). Use part (a) to show that \( E \) is a context-free language.

(Hint: Express \( E \) in a different way.)

7* (Optional) Let \( x \) and \( y \) be strings over some alphabet \( \Sigma \). Say \( x \) is a **substring** of \( y \) if \( y \in \Sigma^* x \Sigma^* \) and say \( x \) is a **major substring** of \( y \) if \( x \) is a substring of \( y \) and \( |x| \geq \frac{1}{2} |y| \).

For any language \( B \), let \( MS(B) = \{ x \mid x \text{ is a major substring of } y \text{ for some } y \in B \} \).

Show that if \( B \) is regular then \( MS(B) \) is regular.