Last time:
- Space complexity
- \( \text{SPACE}(f(n)), \text{NSPACE}(f(n)), \text{PSPACE}, \text{NPSPACE} \)
- Relationship with TIME classes

Today:
- Review \( \text{LADDER}_{\text{DFA}} \in \text{PSPACE} \)
- Savitch’s Theorem: \( \text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n)) \)
- \( \text{PSPACE} \)-completeness
- \( \text{TQBF} \) is \( \text{PSPACE} \)-complete

Posted: Pset 4 solutions, Pset 5
Review: SPACE Complexity

Defn: Let $f: \mathbb{N} \to \mathbb{N}$ where $f(n) \geq n$. Say TM $M$ runs in space $f(n)$ if $M$ always halts and uses at most $f(n)$ tape cells on all inputs of length $n$.

An NTM $M$ runs in space $f(n)$ if all branches halt and each branch uses at most $f(n)$ tape cells on all inputs of length $n$.

$\text{SPACE}(f(n)) = \{B | \text{some 1-tape TM decides } B \text{ in space } O(f(n))\}$

$\text{NSPACE}(f(n)) = \{B | \text{some 1-tape NTM decides } B \text{ in space } O(f(n))\}$

$\text{PSPACE} = \bigcup_k \text{SPACE}(n^k) \quad \text{"polynomial space"}$

$\text{NPSPACE} = \bigcup_k \text{NSPACE}(n^k) \quad \text{"nondeterministic polynomial space"}$

Today: $\text{PSPACE} = \text{NPSPACE}$

Or possibly: $\text{P = NP = coNP = PSPACE}$
Review: \( LADDER_{DFA} \in PSPACE \)

Theorem: \( LADDER_{DFA} \in \text{SPACE}(n^2) \)

Proof: Write \( u \xrightarrow{b} v \) if there’s a ladder from \( u \) to \( v \) of length \( \leq b \).

Here’s a recursive procedure to solve the bounded DFA ladder problem:

\[
\text{BOUNDDED-LADDER}_{DFA} = \{ (B, u, v, b) \mid B \text{ a DFA and } u \xrightarrow{b} v \text{ by a ladder in } L(B) \} 
\]

\( B-L = \) “On input \( \langle B, u, v, b \rangle \) Let \( m = |u| = |v| \).

1. For \( b = 1 \), \textit{accept} if \( u, v \in L(B) \) and differ in \( \leq 1 \) place, else \textit{reject}.

2. For \( b > 1 \), repeat for each \( w \in L(B) \) of length \( |u| \)

3. Recursively test \( u \xrightarrow{b/2} w \) and \( w \xrightarrow{b/2} v \) [division rounds up]

4. \textit{Accept} both accept.

5. \textit{Reject} [if all fail].”

Test \( \langle B, u, v \rangle \in LADDER_{DFA} \) with \( B-L \) procedure on input \( \langle B, u, v, t \rangle \) for \( t = |\Sigma|^m \)

Space analysis:

- Each recursive level uses space \( O(n) \) (to record \( w \)).
- Recursion depth is \( \log t = O(m) = O(n) \).
- Total space used is \( O(n^2) \).
PSPACE = NPSPACE

Savitch’s Theorem: For \( f(n) \geq n \), \( \text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n)) \)

Proof: Convert NTM \( N \) to equivalent TM \( M \), only squaring the space used.

For configurations \( c_i \) and \( c_j \) of \( N \), write \( c_i \xrightarrow{b} c_j \) if can get from \( c_i \) to \( c_j \) in \( \leq b \) steps.

Give recursive algorithm to test \( c_i \xrightarrow{b} c_j \):

\[ M = \text{“On input } c_i, c_j, b \text{ [goal is to check } c_i \xrightarrow{b} c_j \text{] } } \]

1. If \( b = 1 \), check directly by using \( N \)’s program and answer accordingly.
2. If \( b > 1 \), repeat for all configurations \( c_{\text{mid}} \) that use \( f(n) \) space.
3. Recursively test \( c_i \xrightarrow{b/2} c_{\text{mid}} \) and \( c_{\text{mid}} \xrightarrow{b/2} c_j \)
4. If both are true, accept. If not, continue.
5. Reject if haven’t yet accepted.”

Test if \( N \) accepts \( w \) by testing \( c_{\text{start}} \xrightarrow{t} c_{\text{accept}} \) where \( t = \text{number of configurations} \]

\[ = |Q| \times f(n) \times df(n) \]

Each recursion level stores 1 config = \( O(f(n)) \) space.

Number of levels = \( \log t = O(f(n)) \). Total \( O(f^2(n)) \) space.
**PSPACE-completeness**

**Defn:** $B$ is **PSPACE-complete** if

1) $B \in \text{PSPACE}$
2) For all $A \in \text{PSPACE}$, $A \leq_p B$

If $B$ is PSPACE-complete and $B \in \text{P}$ then $\text{P} = \text{PSPACE}$.

**Check-in 18.1**

Knowing that $TQBF$ is PSPACE-complete, what can we conclude if $TQBF \in \text{NP}$?
Check all that apply.

(a) $\text{P} = \text{PSPACE}$
(b) $\text{NP} = \text{PSPACE}$
(c) $\text{P} = \text{NP}$
(d) $\text{NP} = \text{coNP}$

Think of complete problems as the “hardest” in their associated class.
Recall: \( TQBF = \{ \langle \phi \rangle | \phi \text{ is a QBF that is TRUE} \} \)

**Examples:**
\[
\begin{align*}
\phi_1 &= \forall x \exists y [(x \lor y) \land (\overline{x} \lor \overline{y})] \in TQBF \quad \text{[TRUE]} \\
\phi_2 &= \exists y \forall x [(x \lor y) \land (\overline{x} \lor \overline{y})] \notin TQBF \quad \text{[FALSE]}
\end{align*}
\]

**Theorem:** \( TQBF \) is PSPACE-complete

Proof: 1) \( TQBF \in \text{PSPACE} \)

2) For all \( A \in \text{PSPACE}, \ A \leq_P TQBF \)

Let \( A \in \text{PSPACE} \) be decided by TM \( M \) in space \( n^k \).

Give a polynomial-time reduction \( f \) mapping \( A \) to \( TQBF \).

\[
\begin{align*}
&f: \Sigma^* \rightarrow \text{QBFs} \\
f(w) &= \langle \phi_{M,w} \rangle \\
w \in A \text{ iff } \phi_{M,w} \text{ is TRUE}
\end{align*}
\]

Plan: Design \( \phi_{M,w} \) to "say" \( M \) accepts \( w \). \( \phi_{M,w} \) simulates \( M \) on \( w \).
Constructing $\phi_{M,w}$: 1st try

Recall: A tableau for $M$ on $w$ represents a computation history for $M$ on $w$ when $M$ accepts $w$.

Rows of that tableau are configurations.

$M$ runs in space $n^k$, its tableau has:
- $n^k$ columns (max size of a configuration)
- $d(n^k)$ rows (max number of steps)

Constructing $\phi_{M,w}$. Try Cook-Levin method. Then $\phi_{M,w}$ will be as big as tableau.

But that is exponential: $n^k \times d(n^k)$.

Too big! 😞
Constructing $\phi_{M,w}: 2^{\text{nd}}$ try

For configs $c_i$ and $c_j$ construct $\phi_{c_i,c_j,b}$ which “says” $c_i \rightarrow c_j$ recursively.

$$\phi_{c_i,c_j,b} = \exists c_{\text{mid}} \left[ \phi_{c_i,c_{\text{mid}},b/2} \land \phi_{c_{\text{mid}},c_j,b/2} \right]$$

Check-in 18.2

Why shouldn’t we be surprised that this construction fails?

(a) We can’t define a QBF by using recursion.
(b) It doesn’t use $\forall$ anywhere.
(c) We know that $TQBF \not\in P$.

Size analysis:
Each recursive level doubles number of QBFs.
Number of levels is $\log d^{(n^k)} = O(n^k)$.
→ Size is exponential.

Check-in 18.2
Constructing $\phi_{M,w}$: 3rd try

$\phi_{c_i, c_j, b} = \exists c_{mid} \left[ \phi_{c_i, c_{mid}, b/2} \land \phi_{c_{mid}, c_j, b/2} \right]$  

$\forall (c_g, c_h) \in \{(c_i, c_{mid}), (c_{mid}, c_j)\} [\phi_{c_g, c_h, b/2}]$

$\phi_{M,w} = \phi_{c_{start}, c_{accept}, t}$  
$t = d(n^k)$

Size analysis:
Each recursive level adds $O(n^k)$ to the QBF.
Number of levels is $\log d(n^k) = O(n^k)$.
$\rightarrow$ Size is $O(n^k \times n^k) = O(n^{2k})$  

Check-in 18.3
Would this construction still work if $M$ were nondeterministic?
(a) Yes.
(b) No.
Quick review of today

1. $LADDER_{\text{DFA}} \in \text{PSPACE}$

2. Savitch’s Theorem: $\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$

3. $TQBF$ is PSPACE-complete