Last time:
- NP-completeness
- $3SAT \leq_p CLIQUE$
- $3SAT \leq_p HAMPATH$

Today:
- Cook-Levin Theorem: $SAT$ is NP-complete
- $3SAT$ is NP-complete
Quick Review

**Defn:** $B$ is **NP-complete** if

1) $B \in NP$

2) For all $A \in NP$, $A \leq_P B$

If $B$ is NP-complete and $B \in P$ then $P = NP$.

**Importance of NP-completeness**

1) Evidence of computational intractability.

2) Gives a good candidate for proving $P \neq NP$.

To show some language $C$ is NP-complete, show $3SAT \leq_P C$.

or some other previously shown NP-complete language

**Check-in 16.1**

The big sigma notation means summing over some set.

$$\sum_{1 \leq i \leq n} i = 1 + 2 + \cdots + n$$

The big AND (or OR) notation has a similar meaning.

For example, if $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$ are two strings of length $n$, when does the following hold?

$$\left( \bigwedge_{1 \leq i \leq n} x_i = y_i \right) = \text{TRUE}$$

(a) Whenever $x$ and $y$ agree on some symbol.

(b) Whenever $x = y$. 

Check-in 16.1
Theorem: \textit{SAT} is NP-complete

Proof: 1) \textit{SAT} $\in$ \textit{NP} (done)

2) Show that for each $A \in \textit{NP}$ we have $A \leq_{p} \textit{SAT}$:

Let $A \in \textit{NP}$ be decided by NTM $M$ in time $n^{k}$.

Give a polynomial-time reduction $f$ mapping $A$ to \textit{SAT}.

\[ f: \Sigma^{*} \rightarrow \text{formulas} \]
\[ f(w) = \langle \phi_{M,w} \rangle \]
\[ w \in A \text{ iff } \phi_{M,w} \text{ is satisfiable} \]

Idea: $\phi_{M,w}$ simulates $M$ on $w$. Design $\phi_{M,w}$ to “say” $M$ accepts $w$.

Satisfying assignment to $\phi_{M,w}$ is a computation history for $M$ on $w$. 
Defn: An \textbf{(accepting) tableau} for NTM $M$ on $w$ is an $n^k \times n^k$ table representing an \textbf{computation history} for $M$ on $w$ on an accepting branch of the nondeterministic computation.

Construct $\phi_{M,w}$ to “say” $M$ accepts $w$.

$\phi_{M,w}$ “says” a tableau for $M$ on $w$ exists.

$$\phi_{M,w} = \phi_{\text{cell}} \land \phi_{\text{start}} \land \phi_{\text{move}} \land \phi_{\text{accept}}$$
Constructing $\phi_{M,w}$: $\phi_{\text{cell}}$

The variables of $\phi_{M,w}$ are $x_{i,j,\sigma}$ for $1 \leq i,j \leq n^k$ and $\sigma \in \Gamma \cup Q$.

$x_{i,j,\sigma} = \text{TRUE}$ means cell $i,j$ contains $\sigma$.

<table>
<thead>
<tr>
<th>$q_0$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>...</th>
<th>$w_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$q_7$</td>
<td>$w_2$</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$q_{\text{accept}}$</td>
</tr>
</tbody>
</table>

Cell $i,j$ can contain any symbol in $\Gamma \cup Q$.

Check-in 16.2
How many variables does $\phi_{M,w}$ have?
Recall that $n = |w|$.

(a) $O(n)$
(b) $O(n^2)$
(c) $O(n^k)$
(d) $O(n^{2k})$
Constructing $\phi_{M,w}$: $\phi_{\text{start}}$ and $\phi_{\text{accept}}$

$\phi_{M,w}$ “says” a tableau for $M$ on $w$ exists.

$\phi_{M,w} = \phi_{\text{cell}} \land \phi_{\text{start}} \land \phi_{\text{move}} \land \phi_{\text{accept}}$

$\phi_{\text{cell}}$ done ✓

$\phi_{\text{start}} = $

$\phi_{\text{accept}} = \bigvee_{1 \leq j \leq n^k} x_{n^k,j,q,\text{accept}}$
Constructing $\phi_{M,w}: \phi_{move}$

$\phi_{M,w}$ "says" a tableau for $M$ on $w$ exists.

$\phi_{M,w} = \phi_{cell} \land \phi_{start} \land \phi_{move} \land \phi_{accept}$

$\phi_{move} = \bigwedge_{1<i,j<n} \left( \bigvee_{\text{Legal}} \left( x_{i,j-1,R} \land x_{i,j,S} \land x_{i,j+1,T} \land x_{i+1,j-1,V} \land x_{i+1,j,Y} \land x_{i+1,j+1,Z} \right) \right)$

Says that the neighborhood at $i,j$ is legal
Conclusion: \( SAT \) is NP-complete

Summary:
For \( A \in \text{NP} \), decided by NTM \( M \), we gave a reduction \( f \) from \( A \) to \( SAT \):

\[
f: \Sigma^* \rightarrow \text{formulas}
\]

\[
f(w) = \langle \phi_{M,w} \rangle
\]

\( w \in A \) iff \( \phi_{M,w} \) is satisfiable.

\[
\phi_{M,w} = \phi_{\text{cell}} \land \phi_{\text{start}} \land \phi_{\text{move}} \land \phi_{\text{accept}}
\]

The size of \( \phi_{M,w} \) is roughly the size of the tableau for \( M \) on \( w \), so size is \( O(n^k \times n^k) = O(n^{2k}) \).

Therefore \( f \) is computable in polynomial time.
Theorem: 3SAT is NP-complete

Proof: Show SAT \leq_P 3SAT

Give reduction \( f \) converting formula \( \phi \) to 3CNF formula \( \phi' \), preserving satisfiability.
(Note: \( \phi \) and \( \phi' \) are not logically equivalent)

Example: Say \( \phi = ((a \land b) \lor c) \land (\overline{a} \lor b) \)

Tree structure for \( \phi \):

Logical equivalence: \( (A \rightarrow B) \) and \( (A \lor B) \)
\( (A \land B) \) and \( (A \lor B) \)

\( \phi' = ((a \land b) \rightarrow z_1) \land ((\overline{a} \land b) \rightarrow \overline{z}_1) \land ((a \land \overline{b}) \rightarrow \overline{z}_1) \land ((\overline{a} \land \overline{b}) \rightarrow \overline{z}_1) \)
\( \land ((z_1 \land c) \rightarrow z_2) \land ((\overline{z}_1 \land c) \rightarrow \overline{z}_2) \land ((z_1 \land \overline{c}) \rightarrow z_2) \land ((\overline{z}_1 \land \overline{c}) \rightarrow \overline{z}_2) \)
\( \land (z_4) \)

Check-in 16.3

If \( \phi \) has \( k \) operations (\( \land \) and \( \lor \)), how many clauses has \( \phi' \)?

(a) \( k + 1 \) \hspace{1cm} (c) \( k^2 \)

(b) \( 4k + 1 \) \hspace{1cm} (d) \( 2k^2 \)
Quick review of today

1. \textit{SAT} is NP-complete
2. \textbf{3SAT} is NP-complete