Intersection homology was first introduced by Goresky and Macpherson to fix one big problem with the homology of singular spaces: the failure of Poincare duality. To fix this issue, they constructed a new homology theory for triangulated stratified spaces with mild singularities by only considering chains that intersected the singular strata in low enough dimension. They called this homology theory intersection homology and showed that it was equipped with a Poincare duality structure.

As a fortunate accident, it turns out that intersection homology possesses much richer structure than just Poincare duality. For open embeddings, it is possible to define a relative intersection homology for which excision and a Mayer-Vietoris long exact sequence held. Additionally, while intersection homology is not homotopy invariant, it is independent of stratification. These two results combined to make intersection homology a relatively easy to compute measure of the singularities of a space.

Further structure can be found in the intersection homology of complex quasi-projective and projective varieties. In this setting, intersection homology, like regular homology, is equipped with the Lefschetz hyperplane theorem and the hard Lefschetz theorem and that the Hodge decomposition and Hodge signature theorem also holds.

While the above properties show that intersection homology is an extremely interesting geometric invariant of complex algebraic varieties, the goal of this seminar is to study perverse sheaves as they appear in representation theory. The connection of this goal with intersection homology comes with the construction of a sheaf-theoretic version of intersection homology. Following a suggestion by Deligne, Goresky and Macpherson sheafified their definition of intersection homology to get a complex of sheaves, called the $IC$ sheaf of the space, whose hypercohomology gave the intersection homology of the space. The cohomology sheaves of this complex satisfy the support conditions which define the perverse t-structure (after applying a shift). In fact, these $IC$ sheaves furnish all possible examples of
simple perverse sheaves: every simple perverse sheaf on a stratified complex variety is the IC sheaf of a closed stratum (with coefficients in a potentially nontrivial local system defined on the smooth part of the stratum.)

In this talk, we will go through the formal properties that help in computing intersection homology and the IC sheaves that appear in representation theory. Emphasis will be on computational tools and examples and most of the technical details will be left to the references. In section 2, we give definitions and examples of simplicial and singular intersection homology. In section 3, we give an explicit description of the properties of intersection homology mentioned above and show how these properties can be used to compute examples. In section 4, we construct the IC sheaf via Goresky and Macpherson’s sheafification process and via Deligne’s construction. In section 5, we define the constructible derived category and the perverse t-structure and show that the IC sheaf is perverse.

2. Simplicial and Singular Intersection Homology

The main reference for this section is [KW, Chapter 4]. All the definitions and computations can be found there.

We begin with an inductive definition of a topologically stratified space.

**Definition 2.1.** We say that paracompact $X$ is a topologically stratified space of dimension $m$ if there exists a filtration

$$X = X_m \supseteq X_{m-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \emptyset$$

of $X$ by closed subspaces $X_i$ such that for all $x \in X_k - X_{k-1}$, there exists a neighborhood $N_x$ of $x$ in $X$, a compact topologically stratified space

$$L = L_{m-k-1} \supseteq \cdots \supseteq L_0$$

of dimension $m-k-1$ called the link of $x$, and a homeomorphism

$$\phi : N_x \to \mathbb{R}^k \times C(L)$$

with $C(L)$ the open cone on $L$ such that

1. $\phi$ takes $N_x \cap X_{k+i+1}$ homeomorphically onto $\mathbb{R}^k \times C(L_i)$ for $0 \leq i \leq m-k-1$
2. $\phi$ takes $N_x \cap X_k$ homeomorphically onto $\mathbb{R}^k \times \{\text{vertex of the cone}\}$.

We call the connected components of the smooth, locally closed subspaces $X \setminus X_{i-1}$ the strata of $X$.

Up to homeomorphism the link $L$ of $x$ depends only on the stratum that $x$ is in.

To define intersection homology, we demand an additional condition from our spaces.

**Definition 2.2.** We say that $X$ is a topological pseudomanifold of dimension $m$ if it has a topological stratification with $X_{m-1} = X_{m-2}$.

Note that if $X$ is a topological pseudomanifold and $L$ is the link of $x \in X_k \setminus X_{k-1}$, then the homeomorphism $\phi$ takes $\mathbb{R}^k \times C(L_{m-k-2})$ onto $N_x \cap X_{m-1} = N_x \cap X_{m-2}$, which is the image of $\mathbb{R}^k \times C(L_{m-k-2})$. This shows for every point in a pseudomanifold $X$, the associated link is also a pseudomanifold. Hence, we can henceforth forget about stratified spaces that aren’t pseudomanifold.

Here are some examples of topological pseudomanifolds:

**Example 2.3.**
1. The most obvious example is the trivial case of a manifold $M$. Here the stratification is

$$M = X_m \supseteq \emptyset.$$
2. The first nontrivial example would be a cone $C(M)$ on a manifold $M$ of dimension $>1$. This is topologically stratified by

$$C(M) \supseteq \{v\}$$

where $v$ is the vertex of the cone.

3. In similar vein, we can consider the wedge sum $M \vee M$ of a manifold with itself (as long as $M$ has dimension $>1$.)

4. Eventually the main example we will be working with are complex quasi-projective varieties. These will be stratified by closed subvarieties and hence the stratification will be purely even. An important example is that of the affine cone over a (smooth) projective variety.

Intersection homology is defined for topological pseudomanifolds using a function called the perversity that controls the chains under consideration.

**Definition 2.4.** A perversity $p$ is a function $\mathbb{Z}_{\geq 2} \rightarrow \mathbb{Z}$ that satisfies

$$p(2) = 0$$

and

$$p(i + 1) = p(i) \text{ or } p(i) + 1.$$  

There are four examples of perversities that are particularly important.

1. $p(i) = 0$ for all $i$. This is the bottom perversity.
2. $p(i) = i - 2$ for all $i$. This is the top perversity.
3. $p(i) = \lfloor \frac{i}{2} \rfloor - 1$. This is the lower middle perversity.
4. $p(i) = \lceil \frac{i}{2} \rceil - 1$. This is the upper middle perversity.

We now define singular intersection homology as the homology of the complex whose chains are a suitable subset of the standard singular homology chains. Fix a perversity $p$.

**Definition 2.5.** A singular $i$-simplex $\sigma : \Delta_i \rightarrow X$ is said to be $p$-allowable, if

$$\sigma^{-1}(X_{m-k}) \subseteq (i - k + p(k)) - \text{skeleton of } \Delta_i.$$  

Similarly, a Borel-Moore singular $i$-chain is $p$-allowable if each simplex that appears with nonzero coefficient is allowable.

Define $I^{p, BM} S_i(X)$ to be the vector space over $\mathbb{Q}$ consisting of $p$-allowable Borel-Moore singular chains with allowable boundary. Similarly, define $I^{p} S_i(X)$ to be the vector space over $\mathbb{Q}$ consisting of allowable singular chains with compact support with allowable boundary. Note that this does not mean that each simplex in the chain has allowable boundary since cancellation may occur.

We define the intersection homology of $X$ with respect to the perversity $p$ (a priori dependent on the given stratification) as the homology of the complex $I^{p, BM} S_i(X)$ and define the intersection homology with compact supports to be the homology of $I^{p} S_i(X)$.

**Remark.** The perversity function $p$ should be seen as the distance the $p$-allowability condition is from transversality. More precisely, if $p(k) = 0$ for all $k$, then a singular simplex is $p$-allowable if it is transverse to each stratum.

As in the case with ordinary homology, singular intersection homology is useful theoretically but difficult to compute with. So, we now develop a theory of simplicial intersection homology. For this, we need some extra structure on the stratification. Namely, we need a triangulation $T$ on $X$ such that each $X_i$ is a union of simplices. We define $p$-allowability in exactly the same manner as in the singular case and then define the simplicial intersection homology and compactly supported homology as the limit of the homology of the analogous intersection complexes under refinement of the triangulation.
Remark. Unlike the case of ordinary homology, the homology of the simplicial intersection complex associated to a particular triangulation is not isomorphic to the simplicial intersection homology of $X$. But if $T$ is flag-like, i.e., if the intersection of each simplex in $T$ with $X_i$ is a single face, then simplicial intersection homology can be directly computed using $T$.

The simplicial intersection homology and singular intersection homology (and those with compact supports) of $X$ are the same (for a fixed perversity). From now on, we only consider the middle perversity function $p(i)=\lfloor i\rfloor$ and denote the corresponding homology as just intersection homology. We end this section by defining relative intersection homology.

Definition 2.6. Let $X$ be a topologically pseudomanifold and let $U$ be an open subset of $X$. Then, $U$ inherits a stratification from $X$ by defining $U_i = U \cap X_i$. There is an obvious injection $I^pS_i(U) \to I^pS_i(X)$. We define the intersection homology of $X$ relative to $U$ $I^pH_i(X,U)$ to be the homology of the complex $I^pS_i(X)/I^pS_i(U)$.

The relative intersection homology groups satisfy a version of excision. Namely we have the following theorem (see [KW, 4.6] for a further reference)

Theorem 2.7. Suppose $U$ is an open subset of $X$ and $A$ is a subset of $U$ such that the closure of $A$ is contained in $U$ and such that $X\setminus A$ is a topological pseudomanifold. Then, we have an isomorphism

$$I^pH_i(X,U) \cong I^pH_i(X\setminus A,U\setminus A).$$

In particular, the hypotheses of the excision theorem hold if $A$ is a closed subset of $X$ contained in $U$. As a consequence of excision, we have the following Mayer-Vietoris sequence.

Theorem 2.8. Let $U$ and $V$ be open subsets of $X$. Then, we have a long exact sequence

$$\cdots \to I^pH_i(U \cap V) \to I^pH_i(U) \oplus I^pH_i(V) \to I^pH_i(U \cup V) \to I^pH_{i-1}(U \cap V) \to \cdots$$

3. Some Computations

We now compute some examples of intersection homology. The first example is when $X$ is smooth. In this case, the stratification is just $X_{m-i}=\emptyset$ for $i>0$ and the perversity imposes no restriction. Hence the intersection homology and homology of $X$ coincide. Let us now do the simplest nontrivial example.

Example 3.1. Let $X$ be a topological pseudo manifold of dimension $m$ with isolated singularity at $x$. Stratify $X$ by

$$X \supseteq \{x\}$$

and give it a flag-like triangulation (which means that $x$ is a vertex of any simplex that it is contained in.) Then, the allowability condition for $i$-simplices breaks down as follows:

1. $0 \leq i < m - p(m)$: In this dimension range, allowability is equivalent to $x$ not being in $\sigma$ because $i - m + p(m) < 0$. So, $I^pS_i(X) = S_i(X \setminus \{x\})$.
2. $i > m - p(m)$: In this dimension range, allowability imposes no restriction and neither does allowability of the boundary. Hence, $I^pS_i(X) = S_i(X)$.
3. If $i = m - p(m)$, there a chain is always allowable but its boundary being allowable means that its boundary must be contained in $X \setminus \{x\}$.

This shows that

$$I^pH_i(X) = \begin{cases} 
H_i(X) & \text{if } i \geq m - p(m) \\
H_i(X \setminus \{x\}) & \text{if } i < m - p(m) - 1 \\
\text{im}(H_i(X \setminus \{x\}) \to H_i(X)) & \text{if } i = m - p(m) - 1
\end{cases}$$
We apply this example to the case of the cone over a smooth manifold of dimension \( n \). In this case, \( H_i(X) \) is 0 and \( H_i(X \setminus \{x\}) \) is \( H_i(M) \) (by the Kunneth formula). Hence, we get

\[
IH_i(C(M)) = \begin{cases} 0 & \text{if } i \geq n - p(n + 1) \\ H_i(M) & \text{else} \end{cases}.
\]

The next computation is extremely important as it inductively gives the intersection homology of neighborhoods of points in a pseudomanifold. Eventually we will use this to show that the sheafified version of intersection homology satisfies the support conditions that define the category of perverse sheaves.

**Example 3.2.** The intersection homology of a cone. Let \( X \) be a topological pseudomanifold with stratification

\[ X = X_m \supseteq \cdots \supseteq X_0. \]

Consider \( C(X) \), the open cone on \( X \). This is a pseudomanifold with filtration

\[ C(X) = C(X_m) \supseteq C(X_{m-2}) \supseteq C(X_{m-3}) \supseteq \cdots \supseteq C(X_0) \supseteq \{v\} \]

where \( v \) is the vertex of the cone. We compute its intersection homology and Borel-Moore intersection homology. We begin with a Lemma regarding the Kunneth formula for intersection homology. Note that if \( X \) is a pseudomanifold, then so is \( X \times (0, 1) \) (with filtration given by \( X_i \times (0, 1) \)). Then, we have the following lemma (whose proof is essentially the same as in the ordinary homology case).

**Lemma 3.3.**

\[ I^pH_i(X \times (0, 1)) = I^pH_i(X) \]

and

\[ I^{p,BM}H_i(X \times (0, 1)) = I^{p,BM}H_{i-1}(X) \]

with the latter being 0 for \( i = 0 \).

We now use this lemma to prove the following theorem (whose statement and proof are in [KW, 4.7]).

**Theorem 3.4.** If \( X \) is a pseudomanifold of dimension \( m \), then

\[ I^pH_i(C(X)) = \begin{cases} I^pH_i(X) & i < m - p(m + 1) \\ 0 & \text{else} \end{cases} \]

and

\[ I^{p,BM}H_i(C(X)) = \begin{cases} 0 & i \leq m - p(m + 1) \\ I^{p,BM}H_{i-1}(X) & \text{else} \end{cases} \]

**Proof.** We will prove the theorem for ordinary cohomology. For Borel-Moore cohomology, we can either use Poincare duality or use the fact that there is a natural chain map

\[ I^{p,BM}S_i(C(X)) \to I^pS_i(C(X))/I^pS_i(C(X)\setminus\{v\}) \]

which induces an isomorphism on homology and subsequently use the long exact sequence for relative homology along with the Kunneth formula. Alternatively, a similar coning off argument as below can be used (with coning off to infinity replacing coning off to the vertex.)

By essentially the same argument as in the case of a space with isolated singularity, for \( i \leq m - p(m + 1) \), allowable chains must be contained in \( C(X)\setminus\{v\} \). Hence, for \( i < m - p(m + 1) \),

\[ I^pH_i(C(X)) = I^pH_i(X \times (0, 1)) = I^pH_i(X) \]

by the Kunneth formula. Suppose now that \( \sigma : \Delta_i \to X = X \times \{\frac{1}{2}\} \) is an allowable \( i \)-simplex for some \( i \geq m - p(m + 1) \). Representing the \( i + 1 \)-simplex \( \Delta_{i+1} \) by a point \( s \in \Delta_i \) and \( t \in [0, \frac{1}{2}] \) (viewing it as
the closed cone on $\Delta_i$ for example), we define a singular $i+1$-simplex $c\sigma$ by sending $(s,t) \in C(X)$. This is the cone on $\sigma$.

It is fairly easy to see that $c\sigma$ is allowable if $\sigma$ is allowable (see [KW] for example). We extend this coning map to chains by linearity and note that

$$\partial(c\sigma) + c(\partial \sigma) = \sigma$$

for any $i$-chain $\sigma$. Hence, if $\sigma$ is closed and allowable, then it is a boundary of an allowable chain (obviously with allowable boundary). One can show by the same argument as in the case of ordinary homology, that any allowable $i$-chain in $C(X)$ with compact support is equal to one with image in $X \times \{\frac{1}{2}\}$ for some $t$, up to the boundary of an allowable $i+1$-chain. Thus, we see that for $i \geq m - p(m+1)$, we have $I^pH_i(C(X)) = 0$.

\[\square\]

4. Homology with Local Coefficients

As in the case of ordinary homology, we can also define intersection homology with local coefficients. However, there is one big difference: intersection homology works even if the local system is only defined on $X \setminus X_m - 2$ (with $X$ $m$-dimensional here). The reason for this can be seen by looking at how homology with local coefficients is computed.

A $\mathbb{C}$-local system $\mathcal{L}$ on a space $Y$ is a representation of the fundamental groupoid of $Y$ i.e. it is a collection of finite rank $\mathbb{C}$-vector spaces $\mathcal{L}_x$ for each $x \in X$, with an isomorphism $\phi_\gamma : \mathcal{L}_x \to \mathcal{L}_y$ for each homotopy class of path $\gamma$ from $x$ to $y$. The complex of singular chains on $Y$ with coefficients in $\mathcal{L}$ is defined as follows:

1. We define $S_i(Y, \mathcal{L})$ as the vector space generated by $(\sigma, l)$ with $\sigma$ a singular $i$-simplex and $l \in \mathcal{L}_{\sigma(b_i)}$, where $b$ is the barycenter of $\Delta_i$, subject to the relations that $(\sigma, l) + (\sigma, l') = (\sigma, l + l')$.
2. The restriction maps are defined similarly to the constant coefficient case.

$$\partial(\sigma, l) = \sum_i (-1)^i(\sigma_i, (\sigma \circ \gamma_i)(l))$$

where $\sigma_i$ is the face obtained by deleting the $i$th-vertex and $\gamma_i$ is any path from the barycenter of $\Delta_i$ to the barycenter of the $i$th-face. The choice of path is immaterial as all such paths are homotopic.

The homology of this complex is denoted $H_*(Y, \mathcal{L})$. All of these constructions work for intersection homology with one exception, it is sufficient for $\mathcal{L}$ to be defined on $Y_m - Y_m - 2$ if $Y$ is $m$-dimensional. This is because, we only consider chains satisfying the perversity condition and for any such $i$-simplex $\sigma$ in such a chain,

$$\sigma^{-1}(X_{m-2}) \subseteq i - 2\text{ skeleton of } \Delta_i.$$ 

Hence we can still choose a path $\gamma_i$ from the barycenter of $\Delta_i$ to the barycenter of any of its faces that lives within the complement of the $i - 2$-skeleton and all such paths are still homotopic to each other in the $i - 2$-skeleton. Hence, the whole theory carries over without any problem.

Using local coefficient groups defined only on the smooth strata gives us a completely new homology theory, one that often cannot be obtained by using local coefficients defined everywhere. For example, consider $\mathbb{C}$. This is smooth and simply connected. It has no nontrivial local systems defined everywhere and its intersection homology with constant coefficients is just the normal homology of $\mathbb{C}$ which is $1$ in degree $0$ and $0$ in higher degrees. Now, consider the stratification

$$X_2 = \mathcal{A}^1, X_1 = X_0 = \{0\}$$
and let $\mathcal{L}$ be the local system on $\mathbb{C}^*$ whose fundamental group action is given by $n \mapsto a^n$ for $a \neq 1$ (hence $\mathcal{L}$ is nontrivial). Triangulate $\mathbb{A}^1$ with the origin the vertex of any simplex it is contained in. This is a flag-like triangulation compatible with the stratification and can hence be used to compute intersection homology.

The allowability conditions say that any 0 or 1 simplex must live in $\mathbb{C}^*$. Any 2-chain must have its boundary in $\mathbb{C}^*$. One can see that there are then no closed 1-chains or closed 2-chains. Hence, the higher intersection cohomology is 0. However, any 0 simplex $p$ can be seen to be a boundary: if you take a 1-chain constructed from a piecewise linear loop that starts and ends at $p$, then its boundary is $(1-a)p$. Hence, the intersection homology with coefficients in $\mathcal{L}$ is 0 in all degrees, unlike the case of the constant coefficient intersection homology.

**Remark.** Everything stated above can also be done for Borel-Moore chains with perverse restriction. So everything works for Borel-Moore intersection homology as well.

We will in fact need the intersection sheaves defined using these local systems to get a full list of the simple perverse sheaves with respect to a given stratification.

## 5. Some Useful Properties of Intersection Homology

We now give some properties of intersection homology that help in computing it. We will not give the proofs here. Some of the proofs can be found in [GM]. However, some of the proofs become much easier after constructing the sheaf-theoretic framework for intersection homology and will thus be given after developing the latter. In the following, we work only with constant coefficients for simplicity.

1. Intersection homology is independent of stratification i.e. the homology computed using two different stratifications of the same space is the same. Hence, intersection homology is defined for stratifiable spaces and not stratified spaces, and is a topological invariant.

2. Poincare duality: If $p$ is a perversity, then the complementary perversity is $q(i) = i - p(i) - 2$. Let $X$ be a pseudomanifold of dimension $m$. Poincare duality for intersection homology says that there is a perfect pairing

$$I^p H_i(X) \times I^{q,BM} H_{m-i}(X) \to \mathbb{C}$$

In particular, this holds when $p$ is the lower middle perversity and $q$ is the upper middle perversity. If $X$ is a complex quasi-projective variety, it is stratified by even dimensional spaces and hence the lower and upper middle perversities give the same homology groups. Hence, in this case, we have duality between $I^p H_i(X)$ and $I^{p,BM} H_{m-i}(X)$. This case of Poincare duality will follow from Verdier duality in the constructible derived category of $X$.

3. The Artin Vanishing Theorem: the Artin Vanishing Theorem for intersection homology generalizes that for cohomology constructible sheaves on affine complex varieties. Suppose $X$ is an affine complex variety. Then, $X$ is a pseudomanifold and hence intersection homology can be defined for $X$. Whenever we speak about complex varieties, we will use the lower middle perversity and omit the symbol $p$. In this case, the theorem states that

$$IH_i(X) = 0 \text{ if } i > \dim(X)$$

and

$$I^{BM} H_i(X) = 0 \text{ if } i < \dim(X)$$

where, here, we use the complex dimension. The second statement obviously follows from the first and vice versa. The second statement can be proved by using the Artin Vanishing Theorem for constructible sheaves on smooth spaces and a spectral sequence argument, once we set up the sheaf-theoretic framework.
4. The Lefschetz Hyperplane Theorem: We state a weaker version of the full hyperplane theorem here. A stronger one will be given with proof once we develop some sheaf theory. Let $X$ be a projective variety and let $X_1$ be a hyperplane section of $X$ transverse to each strata in some stratification of $X$. Then, the natural map,

$$IH_i(X_1) \to IH_i(X)$$

is an isomorphism for $i < \dim(X) - 1$ and is injective for $i = \dim(X) - 1$. A similar statement can be made for $i \geq \dim(X) + 1$ using Poincare duality.

Remark. During the lecture, I was trying to globalize the sheaf-theoretic proof that used the Artin Vanishing theorem but what I said was definitely not correct.

5. There is a version of the Hard Lefschetz theorem and the Hodge decomposition for the intersection homology of projective varieties but I will omit them for now.

6. Normalization. Suppose $X$ is a complex quasi-projective variety and $\tilde{X}$ is its normalization. Then, $IH_i(X) \cong IH_i(\tilde{X})$. In particular, this implies that intersection homology for curves can be computed as homology on smooth curves.

7. Cohomological Decomposition Theorem: This is the first instance where the theory of intersection homology in local coefficients becomes important and is the main reason why we do need to allow local coefficients defined only on the smooth strata. This is a global version of the decomposition theorem for perverse sheaves and so we omit the proof for now.

Let $f : X \to Y$ be a proper map of complex quasi-projective varieties, with $X$ of dimension $m$. Then, for each $q$, there is a finite list $EV_q$ of pairs $(S, L)$, where $S$ is a closed subvariety of $Y$ and $L$ is a local system defined on a dense open smooth subvariety $S_0$ of $S$, such that

$$I^{BM}H_{m-i}(X, \mathbb{C}) \cong \bigoplus_{q \geq 0, (S, L) \in EV_q} I^{BM}H_{\dim(S)-i+q}(S, L).$$

This theorem has limited utility unless one can figure out which $S$ and $L$ appear in the right hand side. This is easier done by looking at the sheaf-theoretic decomposition theorem once the language of perverse sheaves is fully developed, although, except in the case of semismall maps, it is still a difficult task.

### 6. Sheaf-Theoretic Intersection Homology

We now develop the sheaf-theoretic framework for intersection homology. We now work only with complex algebraic quasi-projective varieties and the (lower) middle perversity. Let $X$ be a quasi-projective variety of complex dimension $n$ equipped with a stratification

$$X = X_{2n} \supseteq X_{2n-2} \supseteq X_{2n-4} \supseteq \cdots \supseteq X_0$$

by closed subvarieties $X_i$. We start by defining a complex of sheaves on $X$. For now, we work only with intersection sheaves defined with respect to the constant sheaf.

**Definition 6.1.** Define a presheaf of singular $i$-chains, $S^i$ as

$$S^i(U) = S^{BM}_{n-i}(U)$$

where for $V \subseteq U$, the restriction map is defined as follows:

1. Let $\sigma$ be a singular simplex on $U$. We define a set $T_{\sigma, V}$. First set $T_{\sigma, V} = \{\sigma\}$ if the support of $\sigma$ is in $V$. If this is not the case, then barycentric subdivide $\sigma$, and add any simplex in the subdivision with support inside $V$ to $T_{\sigma, V}$. If $\tau$ is a simplex in the subdivision with support not inside $V$, subdivide again and repeat the process. Continue this process indefinitely, to get a well-defined, possibly infinite set $T_{\sigma, V}$ of singular simplices with support in $V$. 

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2. Define the restriction of \( \sigma \) to \( V \) to be

\[
\sum_{\tau \in T_{\sigma,V}} \tau.
\]

Extend this definition to all Borel-Moore singular chains by linearity (which can be done by the locally finite hypothesis). It is easy to see that the resulting chain will also be locally finite in \( V \).

This defines a presheaf on \( X \) and one can check that it is actually a sheaf. Additionally, the restriction map preserves the allowability conditions and hence sends the chains contained in \( I^{BM}S_{n-i}(U) \) to \( I^{BM}S_{n-i}(V) \) if \( V \subseteq U \). Hence, we also get a well-defined sheaf by setting \( I^{BM}\mathcal{C}^i(U) = I^{BM}S_{n-i}(U) \), with the above restriction map. The differential commutes with restriction, and hence we get the singular complex of sheaves \( S^* \) and the intersection complex of sheaves \( I^{BM}\mathcal{C}^* \) (which we call the IC-sheaf).

**Remark.** It is also possible to work directly with cohomology by setting \( I^{BM}\mathcal{C}^i = IS_{n-i,n} \) but since we’ve done all our calculations with intersection homology above, we will continue using the definition via homology.

Our first goal is to show that the hypercohomology of \( I^{BM}\mathcal{C}^* \) (respectively \( S^* \)) is \( IH^{BM}S_{n-*}(X) \) (resp. \( H^{BM}_{n-*}(X) \)) and the hypercohomology with compact supports is \( IS_{n-*}(X) \) (resp. \( S_{n-*}(X) \)). These are clearly the complexes obtained by applying the global sections functor or the compactly supported global sections functor. Hence, it suffices to show that the \( I^{BM}\mathcal{C}^* \) and \( S^* \) are acyclic for these functors.

We do this by showing that they are c-soft.

**Definition 6.2.** A sheaf \( \mathcal{F} \) on a topological space \( X \) is c-soft if for every compact \( K \subseteq X \), the restriction map

\[
\Gamma(\mathcal{F},X) \to \Gamma(\mathcal{F}|_K,K)
\]

is surjective, where \( \mathcal{F}|_K = i_K^{-1}\mathcal{F} \) with \( i_K \) the inclusion of \( K \) into \( X \).

The reason this definition is important is that on locally compact spaces (such as complex algebraic varieties), c-soft sheaves are acyclic for \( \Gamma,\Gamma_c \) (global sections and global sections with compact support) Hence, to show what we want, it suffices to prove the following proposition.

**Proposition 6.3.** The sheaves \( I^{BM}\mathcal{C}^i \) and \( S^i \) are c-soft on \( X \).

**Proof.** We give the proof for \( I^{BM}\mathcal{C}^i \). The proof for \( S^i \) is completely analogous. Let \( K \) be a compact subspace of \( X \) and suppose \( \alpha \) is a global section of \( I^{BM}\mathcal{C}^i|_K \). Then, there exists some open subset \( U \) of \( K \), such that \( \alpha \) is represented by a locally finite intersection i-chain \( i_U \). Now for each \( x \in K \), we can fine a neighborhood \( V_x \) of \( x \in X \) such that meets the support of only finitely many of the simplices appearing in \( \alpha_U \). Cover \( K \) by finitely many of the \( V_x \) to get an open neighborhood \( V \) of \( K \). Then, \( V \) intersects the support of only finitely many singular simplices \( \{ \sigma_j : j \in F \} \) appearing in \( \alpha|_U \) (this set is the union of the sets for each \( V_x \)). Define

\[
\tilde{\alpha} = \sum_{j \in F} c_{\sigma_j} \sigma_j \in I^{BM}\mathcal{C}^i(U)
\]

where \( c_{\sigma_j} \) is the coefficient of \( \sigma_j \) in \( \alpha_U \). Then,

\[
\tilde{\alpha}|_V = \alpha|_V
\]

as the other simplices in \( \alpha_U \) have supports disjoint from \( V \). Now, \( \tilde{\alpha} \) is a finite singular intersection chain and hence extends to a singular chain on \( X \). Since \( \tilde{\alpha} \) represents \( \alpha \) in \( I^{BM}\mathcal{C}^i|_K \), we see that the map \( \Gamma(X,I^{BM}\mathcal{C}^i) \to \Gamma(K,I^{BM}\mathcal{C}^i|_K) \) is surjective as desired.

As a consequence we have the desired result:

**Corollary 6.4.** The hypercohomology of \( I^{BM}\mathcal{C}^* \) (respectively \( S^* \)) is \( IH^{BM}S_{n-*}(X) \) (resp. \( H^{BM}_{n-*}(X) \)) and the hypercohomology with compact supports is \( IS_{n-*}(X) \) (resp. \( S_{n-*}(X) \)).
Remark. Note that we can do the above construction of the $IC$-sheaves with respect to any local system $L$ defined on $X \setminus X_{m-2}$. This construction is completely analogous to the one associated to the constant sheaf, using iterated barycentric subdivision to define the restriction maps. These complex of sheaves will be denoted $IC_*^L$. The same proof as above generalizes to show that this complex is $c$-soft and hence acyclic for $\Gamma, \Gamma_*$. Thus, we get an analogous result for the hypercohomology and hypercohomology with compact supports of this complex.

We end this talk by showing that the cohomology sheaves of the $IC$-sheaves constructed above with respect to the constant sheaf satisfy certain restrictions on the dimensions of their supports. Later in the seminar, we will see that these restrictions imply that the $IC$-sheaves satisfy the famous support conditions that form one half of the definition of the category of perverse sheaves on $X$. In fact, these restrictions are a stronger version of the support conditions that imply that $IC^*$ is a simple perverse sheaf.

**Theorem 6.5.** Let $X$ be a quasi-projective complex variety of complex dimension $n = \frac{m}{2}$ and let

$$X = X_m \supseteq X_{n-2} \supseteq \cdots \supseteq X_0$$

be a stratification for $X$. Let $IC^*$ be the $IC$-sheaf associated to the constant sheaf constructed with respect to this stratification. Then, for positive $i$,

$$\text{supp}(H^{-i}(IC^*)) \subseteq X_{2(i-1)}$$

unless $i = n$ and $H^{-n}(IC^*)$ is supported everywhere. Additionally, for nonpositive $i$, $H^{-i}(IC^*) = 0$.

**Proof.** Suppose $i \neq n$. We first prove that for $x \in X_{2i} \setminus X_{2(i-1)}$, the stalk $H^{-j}(IC^*)_x$ vanishes for $j \leq i$. Since taking stalks commutes with taking cohomology, we thus need to prove that for $x \in X_{2i} \setminus X_{2i-2}$, $H^{-j}(IC^*_x) = 0$ for $j \leq i$. Now,

$$H^{-j}(IC^*_x) = \lim_{U \ni x} I^{BM}_{n+j} H_n(U).$$

Now, by the topological stratification of $x$, we have a neighborhood of $x$ homeomorphic to $\mathbb{R}^{2i} \times C(L)$, where $L$ is a topological pseudomanifold of dimension $m - 2i - 1$, with $x$ the vertex of the cone. Hence, we actually have a basis of neighborhoods all homeomorphic to $\mathbb{R}^i \times C(L)$. Thus, as we work with the middle perversity and $p(m - 2i) = n - i - 1$,

$$H^{-j}(IC^*_x) = I^{BM}_{n+j} H_{n+j} (\mathbb{R}^{2i} \times C(L)) = I^{BM}_{n+j} H_{n+j-2i} (C(L)).$$

Since $j \leq i$, we have

$$m - 2i - 1 - p(m - 2i) = n - i \geq n + j - 2i$$

and thus, we see by Theorem 3.4 that

$$H^{-j}(IC^*_x) = 0.$$

Note that the above works as long as $i \neq n$, and even holds for negative $j$. To finish the proof, we need to compute the stalks of $H^i(IC^*)$ at points on $X \setminus X_{2m-2}$ and show that for $i = -n$, the stalks are nonzero and for $i \neq -n$, the stalks are 0. Note that each point $x \in X \setminus X_{2m-2}$ has a basis of neighborhoods homeomorphic to $\mathbb{R}^{2m}$. Hence,

$$H^i(IC^*)_x = H^i(IC^*_x) = I^{BM}_{n-i} (\mathbb{R}^{2n})$$

which is nonzero if and only if $i = -n$. This finishes the proof of the theorem.

**References**

[GM] M. Goresky and R. Macpherson. Intersection homology theory, II.