CHARACTER FORMULAS FOR SIMPLE \(D\)-MODULES ON THE CYCLIC QUIVER VIA CYCLOTOMIC RATIONAL CHEREDNIK ALGEBRAS

Abstract. In this article, we study a series of functors introduced by Montarani, in [Mon], from the category of \(D\)-modules on the quiver varieties associated to the cyclic quiver with \(r\) vertices to the category of modules of the rational Cherednik algebra associated to the complex reflection group \(G(r,1,n)\). We compute the images of simple \(D\)-modules under these functors and use this data to compute some character formulas for the \(D\)-modules.

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1. Introduction

2. Preliminaries

The main objects of study in this paper are \(D\)-modules on the cyclic quiver with \(r\) vertices and the rational Cherednik algebra associated to \(G(r,1,n)\). We begin by defining these objects.

2.1. \(D\)-modules on the cyclic quiver.

**Definition 2.1.** The cyclic quiver \(Q_r\) is the quiver with \(r\) vertices \(\{0,\ldots, r-1\}\) and one arrow from \(i\) to \(i+1\) (where this index for the vertex set is taken mod \(r\)).

The arrows for \(Q_r\) thus form a single cycle, hence giving the quiver its name.

**Definition 2.2.** A representation of \(Q_r\) is the data of a vector space \(V_i\) for each vertex \(i\) and a collection of linear maps \(\rho_i : V_i \to V_{i+1}\). More generally a representation of a quiver is the data of a vector space for each vertex of the quiver and a linear map between them for each arrow in the quiver.
We write $\text{Rep}(Q_r, \alpha)$ for the space of all representations of $Q_r$ with fixed dimension vector $\alpha = (\alpha_0, \ldots, \alpha_{r-1})$. In this paper, we will primarily consider the equidimensional case where $\alpha = (N, \ldots, N)$ and will use the notation $\text{Rep}(Q_r, N)$ to denote the corresponding space of quiver representations. Concretely,

$$\text{Rep}(Q_r, \alpha) = \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} \text{Hom}(V_i, V_{i+1})$$

where each $V_i$ is a vector space of dimension $v_i$.

A $D$-module over $\text{Rep}(Q_r, \alpha)$ is a module over the algebra of polynomial differential operators on this space. Here is a more concrete description of this algebra

**Definition 2.3.** For a vector space $V$ with basis $y_1, \ldots, y_n$ and dual basis $x_1, \ldots, x_n$, the algebra of polynomial differential operators in $V$ is

$$D(V) = \mathbb{C}(y_1, \ldots, y_n, x_1, \ldots, x_n)/([y_i, x_i] = 1).$$

Note that $\prod_i GL(\alpha_i)$ acts on the $\text{Rep}(Q, V)$ by linear automorphisms. Hence, by differentiating, we get a map from $\prod_i \mathfrak{gl}(\alpha_i)$ to vector fields on $\text{Rep}(Q, \alpha)$ and hence to $D\text{Rep}(Q, \alpha)$.

**Definition 2.4.** We say that a $D$-module $M$ on $\text{Rep}(Q, \alpha)$ is equivariant if it is equipped with an action of $\prod_i GL(\alpha_i)$ that integrates the action via vector fields of $\prod_i \mathfrak{gl}(\alpha_i)$.

In this paper, we look at a very special class of equivariant $D$-modules, those that come about by pushing forward the functions on nilpotent orbits.

**Definition 2.5.** Let $O$ be a $\prod_i GL(\alpha_i)$-orbit in $\text{Rep}(Q_r, \alpha)$. Then,

$$\text{IC}(O)$$

is the unique irreducible equivariant $D$-module on $\text{Rep}(Q_r, \alpha)$ that is supported on the closure of $O$ and whose restriction to $O$ is $O_O$.

In [Lus], it is explained that these $D$-modules form a list of all irreducible holonomic equivariant $D$-modules on $\text{Rep}(Q_r, \alpha)$. The goal of this paper is derive character formulas for these modules. Unfortunately, a concrete description of these modules is hard to obtain so we work with some related $D$-modules: $i_* O_O$, where $i$ is the inclusion of $O$ into $\text{Rep}(Q_r, \alpha)$. A description of these modules can be found in any introductory text on $D$-modules (where it would more properly be called $H^0(i_* O_O)$). The main properties that we will use are

1. $\mathbb{C}[O] \subseteq i_* (O_O)$
2. $i_* (O_O)$ has a unique irreducible submodule $\text{IC}(O)$. All other composition factors of $i_* (O_O)$ are supported on the complement of $O$ in $\overline{O}$.
3. If $O$ is a closed orbit, then $i_* (O_O) = \text{IC}(O)$.

### 2.2. Nilpotent Orbits and $r$-partitions.

Let $\{V_i : i \in \mathbb{Z}/r\mathbb{Z}\}$ be the vector spaces at the vertices of $Q_r$ in $\text{Rep}(Q_r, \alpha)$. A representation $f$ of the quiver is a collection of maps $f_i : V_i \to V_{i+1}$ for each $i$. Taking an $r$-fold composition $f^r$ gives maps $V_i \to V_i$ for each $i$.

**Definition 2.6.** The nilpotent cone $N \subseteq \text{Rep}(Q_r, \alpha)$ consists of all representations $f$ such that $f^r$ is nilpotent.

For $f$ to be nilpotent, it clearly suffices for any one map $V_i \to V_i$ determined by $f^r$ to be nilpotent. Note that orbits under the $GL$-action on $\text{Rep}(Q_r, \alpha)$ are split into two sets: those consisting entirely of nilpotent representations and those consisting entirely of non-nilpotent representations. In this paper, we restrict ourselves to nilpotent orbits. Here is an explicit description of these orbits from [Lus].

A nilpotent orbit is determined by a list of pairs of positive integers and elements of $\mathbb{Z}/r\mathbb{Z}$. To such a list $\{(n_i, a_i)\}$, a representative of the orbit is described by the maps

$$v_i^1 \mapsto v_i^2 \mapsto \cdots \mapsto v_i^{n_i} \mapsto 0$$
for each $i$, with the vectors $\nu^i_j$ forming a basis for the vector space $V_{j+1}$. Two such lists give representations in the same orbit if and only if they are the same under reordering.

To any such orbit, we associate a combinatorial object called an $r$-partition.

**Definition 2.7.** An $r$-partition of a non-negative integer $n$ is an $r$-tuple $\lambda = (\lambda^0, \ldots, \lambda^{r-1})$, where each $\lambda^i$ is a partition and the sum of the sizes of $\lambda^i$ equals $n$.

The $r$-partition $\lambda$ associated to an orbit $O = \{(a_1, a_2)\}$ is defined as follows. To define $\lambda^i$, we pick up all $n_j$ such that $a_j + n_j = i \mod r$. Concretely, we want the ending vector in a string to be in $V_i$. Ordering these in non-decreasing order gives us a partition, which is $\lambda^i$. The size of the $r$-partition is $\sum_i a_i$. This $r$-partition clearly determines the orbit $O$. Let us denote the associated $r$-partition by $\lambda(O)$.

### 2.3. Gan-Ginzburg Algebra of $Q_r$ and the rational Cherednik algebra $H_{t,c,d}(G(r, 1, n))$

We recall the definitions of the Gan-Ginzburg algebra ([GG], [Mon]) and the rational Cherednik algebra of $G(r, 1, n)$. ([DG], [EG].]

Let $B := \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} \mathbb{C} e_i$ be the semisimple algebra over $\mathbb{C}$ in which the $e_i$ are orthogonal idempotents. So, $B$ is isomorphic to $\mathbb{C}^r$ as an algebra. Let $E$ be the $\mathbb{C}$ vector space whose basis is given by the set of all arrows in $Q_r$, the doubled quiver associated to $Q_r$. More concretely, $E = \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} \mathbb{C} f_i \oplus \mathbb{C} f_i^*$, where $f_i$ is the arrow from $i$ to $i + 1$ and $f_i^*$ is the opposite arrow. Then, $E$ is naturally a $B$-bimodule. Fixing a number $n$, we now construct some new objects from $B$ and $E$:

\[ B = B^\otimes n, \mathcal{E} = B^{\otimes (l-1)} \otimes E \otimes B^{\otimes (n-l)} \]

and

\[ \mathcal{E} = \bigoplus_{i=1}^n \mathcal{E}_i. \]

The Gan-Ginzburg algebra is a quotient of the smash product $T_B \mathcal{E} \rtimes S_n$, where $S_n$ acts on $T_B \mathcal{E}$ via the obvious action on $E$. To define the relations, we need to define certain projectors in $T_B \mathcal{E}$.

Note that $T_B(E)$ is the path algebra $\mathbb{C}[Q_r]$. Thus, for any path $p$ in the doubled quiver, we have the corresponding element of $T_B \mathcal{E}$

\[ (p)_l = 1 \otimes \cdots \otimes q \otimes 1 \otimes \cdots \otimes q \]

where $p$ is put in the $l$th position. For any pair $l, m \in [1, n]$, we define the projector in $T_B \mathcal{E}$

\[ P_{l,m} = \sum_{i \in \mathbb{Z}/r\mathbb{Z}} (e_i)_l (e_i)_m. \]

**Definition 2.8.** Let $\nu, \lambda$ be complex numbers, and set $\lambda = \sum_{i \in \mathbb{Z}/r\mathbb{Z}} \lambda_i e_i \in B$. The Gan-Ginzburg algebra $A_{n,\nu,\lambda}(Q_r)$ is the quotient of the semidirect product $T_B \mathcal{E} \rtimes S_n$ by the relations

1. For any $l \in [1, n]$,

\[ \sum_{i \in \mathbb{Z}/r\mathbb{Z}} [(f_i)_l, (f_i^*)_l] = (\lambda)_l + \nu \sum_{m \neq l} P_{l,m} s_{l,m} \]

where $s_{l,m}$ is the transposition swapping $l$ and $m$ in $S_n$.

2. For any $a, b$ arrows in $Q_r$, $l \neq m$,

\[ [(a)_l, (b)_m] = \begin{cases} 0 & a \neq b^* \\ \nu (e_{h(a)})(e_{t(a)}) s_{l,m} & a = b^* \text{ and } b \in Q \\ \nu (e_{h(a)})(e_{t(a)}) s_{l,m} & a = b^* \text{ and } a \in Q \end{cases} \]

where $h(a), t(a)$ are respectively the head and tail of the arrow $a$.

The Gan-Ginzburg algebra can be defined for an arbitrary quiver. For $Q_r$, this algebra turns out to be isomorphic to the rational Cherednik algebra for $G(r, 1, n)$. Recall first that the complex reflection group $G(r, 1, n)$ is the wreath product, $(\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$. Let us denote by $x$, the generator of $\mathbb{Z}/r\mathbb{Z}$ and by $z_i$, the generator in position $i$. Let us denote by $\zeta$ a primitive $r$th root of unity in $\mathbb{C}$. 


$G(r, 1, n)$ has a natural reflection representation on an $n$-dimensional complex vector space $V$ given by permutation matrices and diagonal matrices with $r$th roots of unity on the diagonal. Let $Y_1, \ldots, Y_n$ be a basis for $V$ and $X_1, \ldots, X_n$ the dual basis of $V^\ast$.

**Definition 2.9.** ([DG][1.4] up to a scaling factor chosen to fit better with [Mon] conventions) Let $t, c, d_0, \ldots, d_r$ be complex numbers such that $d_0 + \cdots + d_r = 0$. The rational Cherednik algebra $H_{t,c,d}(G(r, 1, n))$ (alternatively denoted by $H_{t,c,d}(n)$) is the $C$-algebra generated by $Y_1, \ldots, Y_n, X_1, \ldots, X_n$ and $G(r, 1, n)$ subject to the relations

$$[Y_i, Y_j] = 0 = [X_i, X_j]$$

$$wY_iw^{-1} = w(Y_i), wX_iw^{-1} = w(X_i)$$

$$X_iY_i = Y_iX_i + t - c \sum_{j \neq i} \sum_{l=0}^{r-1} \zeta^l s_{ij} z_i^{-l} - \sum_{j=0}^{r-1} (d_j - d_{j-1}) e_{ij}$$

and for $i \neq j$,

$$X_iY_j = Y_jX_i + c \sum_{l=0}^{r-1} \zeta^{-l} z_i^l s_{ij} z_i^l$$

where

$$e_{ij} = \frac{1}{r} \sum_{l=0}^{r-1} \zeta^{-lj} z_i^l.$$ 

Generally, for a quiver of ADE type, the Gan-Ginzburg algebra is Morita equivalent to the associated symplectic reflection algebra ([GG]). In the case of the cyclic quiver, this Morita equivalence is actually an isomorphism. Here is a description of this isomorphism. We leave the proof as an exercise in tracing out the Morita equivalence maps in [GG].

$$Y_i \mapsto \sum_j (f_j)_i, X_i \mapsto \sum_j (f_j^*_i)_i, \frac{1}{r} \sum_j \zeta^{-lj} z_i^l \mapsto (e_i)_i,$$

where $e_i$ is the idempotent in $B$.

### 2.4. PBW Basis for the rational Cherednik algebra

An important property for the rational Cherednik algebra is the existence of the PBW basis.

**Proposition 2.10.** As a vector space,

$$H_{t,c,d}(n) = \mathbb{C}[Y_1, \ldots, Y_n] \otimes \mathbb{C}[X_1, \ldots, X_n] \otimes \mathbb{C}[G(r, 1, n)].$$

In particular, this implies that the algebra generated by the $X_i$ and $G(r, 1, n)$, namely

$$P(n) := \mathbb{C}[X_1, \ldots, X_n] \rtimes G(r, 1, n)$$

is a subalgebra of $H_{t,c,d}(n)$.

### 2.5. Category $\mathcal{O}$ for the rational Cherednik algebra

An important notion associated to rational Cherednik algebras is that of category $\mathcal{O}$. The full category of modules over the rational Cherednik algebra is complicated so we look at a subcategory called category $\mathcal{O}$. To define this subcategory, we need to first define a special element of the algebra.

$$h := \sum_i X_iY_i + Y_iX_i.$$
Definition 2.11. Category $\mathcal{O}$ or $\mathcal{O}^+$ for the rational Cherednik algebra of $G(r, 1, n)$ is the full subcategory of the category of modules consisting of those modules which and have locally nilpotent action of $\mathbb{C}[Y_1, \ldots, Y_n]$ and are finitely generated over $\mathbb{C}[X_1, \ldots, X_n]$ (or equivalently over the whole Cherednik algebra under the local nilpotency condition.) Similarly, category $\mathcal{O}^-$ is defined by swapping the roles of $X$ and $Y$.

Category $\mathcal{O}^-$ is a lowest weight category. We won't go into too many details regarding the definition of such a category. What we will use is the fact that every object in this category has finite length and irreducibles in this category are in bijection with irreducible representations of $G(r, 1, n)$. We recall the construction of this bijection.

Let $\lambda$ be a finite dimensional representation of $G(r, 1, n)$. We extend the action of the group to an action of $P(n)$ by specifying $X_i$ to act by 0. $P(n)$ is a subalgebra of $H_{t,c,d}(n)$ so we can induce this representation up.

Definition 2.12. The Verma Module $M(\lambda)$ associated to $\lambda$ is

$$M(\lambda) := \text{Ind}_{P(n)}^{H_{t,c,d}(n)} \lambda.$$ 

By the PBW property of the Cherednik algebra, as a vector space

$$M(\lambda) = \mathbb{C}[Y_1, \ldots, Y_n] \otimes \lambda.$$ 

This is clearly finitely generated over $\mathbb{C}[Y_i]$. Additionally, the commutation relations between the $X$’s and the $Y$’s, and the fact that $X_i$ kills $\lambda$, show that the $X_i$ act locally nilpotently on $M(\lambda)$. Hence, $M(\lambda) \in \mathcal{O}^-$. Additionally, if $\lambda$ is irreducible, then $M(\lambda)$ has a unique irreducible quotient because all proper $H_{t,c,d}(n)$ submodules must not contain any vectors from $\lambda$. Let us denote the unique irreducible by $L(\lambda)$.

Proposition 2.13. The $L(\lambda)$ form a list of all irreducibles in $\mathcal{O}^-$. Distinct $\lambda$ and $\mu$ give nonisomorphic $L(\lambda)$ and $L(\mu)$. Hence, the irreducibles in $\mathcal{O}^-$ are in bijection with irreducible representations of $G(r, 1, n)$.

Proof. The proof is a standard fact in the representation theory of rational Cherednik algebras.

Given an arbitrary module $M$ in $\mathcal{O}^-$, an important task is to figure out the irreducible composition factors of $M$. By the local nilpotency condition, $M$ is generated by its lowest weight vectors.

Definition 2.14. We say that $v \in M$ is a lowest weight vector if it is killed by every $X_i$.

The space of all lowest weight vectors in $M$ is a finite dimensional representation of $G(r, 1, n)$. Suppose this representation breaks down into $\lambda_1 \oplus \cdots \lambda_n$. Then, the composition factors of $M$ are precisely $L(\lambda_i)$.

We end this subsection by giving a combinatorial description of the irreducible representations of $G(r, 1, n)$. Irreducibles are in bijection with $r$-partitions. Here is the construction. Suppose we have an $r$-partition of $n$ $(\lambda^1, \ldots, \lambda^r)$. Let $n_i$ be the size of $\lambda^i$. We first construct irreducible representations of $G(r, 1, n_i)$ by taking $V(\lambda^i)$, the irreducible Specht module for $S_{n_i}$ and tensoring with the $i$th 1-dimensional representation of $\mathbb{Z}/r\mathbb{Z}$. Let us denote this representation as $V(\lambda^i) \otimes L_i$.

Definition 2.15. The irreducible $S(\lambda)$ of $G(r, 1, n)$ is

$$\text{Ind}_{\prod_i G(r, 1, n_i)}^{G(r, 1, n)} (\bigotimes_i V(\lambda^i) \otimes L_i).$$

The proof that these are all the irreducibles of $G(r, 1, n)$ is extremely standard. We gave the construction here because the explicit description will be useful for us later.

2.6. Schur-Weyl duality for $G(r, 1, n)$. Note that $V^{\otimes n}$ has a commuting action of both $\prod_i GL(V_i)$ and $G(r, 1, n)$. Polynomial irreducible representations of the former are labeled by $r$-partitions where the $i$th partition has $\leq \alpha_i$ rows. Irreducibles for the latter are labeled by $r$-partitions of total size $n$. Let us denote the irreducibles by $V(\lambda)$ and $S(\lambda)$ respectively. The following describes how $V^{\otimes n}$ is decomposed into irreducible representations of $G(r, 1, n) \prod_i GL(V_i)$. 


Proposition 2.16. As a representation of $G(r, 1, n) \prod_i GL(V_i)$,

$$V^{\otimes n} = \bigoplus_{\lambda} V(\lambda) \boxtimes S(\lambda)$$

where the sum is over all $r$-partitions of total size $n$ and such that the $i$th partition has $\leq \alpha_i = \dim(V_i)$ rows.

Proof. The proof uses classic Schur-Weyl duality for $GL(V_i)$ and $S_{\lambda}$, along with the description of $S(\lambda)$ in the previous section. It is standard so we will not elaborate.

\[\square\]

2.7. Fourier Transforms. The Fourier transform is an anti-autoequivalence $F$ of the category of equivariant $D$-modules. For a $D$-module $M$, the action of differential operators on $F(M)$ is obtained by sending a basis for $\text{Rep}(Q, \alpha)$ to its dual basis and vice versa and sending $g \in \prod_i GL(\alpha_i)$ to $g^{-1}$. Note that $F^2$ is isomorphic to the identity.

As an example, the Fourier transform sends the $D$-module $C[\text{Rep}(Q, \alpha)]$ to $i_n(O)$ where $O$ is the orbit corresponding to the 0 representation.

2.8. Montarani’s Functor. In [CEE], the authors constructed a family of functors from equivariant $D$-modules on the Jordan quiver to modules of the rational Cherednik algebra for the symmetric groups. In [Mon], Montarani extended the definition of this functor to other quivers. In the case of the cyclic quiver, we get a family of functors to rational Cherednik algebras associated to $\text{GL}(r, 1, n)$.

Let $\chi_i$ be a character of $\mathfrak{gl}(v_i)$ and let $\chi$ be the resulting character of $\mathfrak{g}$. Let $V = \oplus_{i \in \mathbb{Z}/r\mathbb{Z}} V_i$ be the representation space of the quiver.

Definition 2.17. We define the functor $F_{n, \lambda} : D^G(\text{Rep}(Q, v)) \to H_{i, c, d}(n)\text{-mod}$ by sending

$$M \mapsto (M \otimes V^{\otimes n})^g$$

As yet, this functor only takes values in vector spaces. To upgrade it to a functor to Cherednik algebras, we need to describe the action of $Y_i, X_i$ and elements of $G(r, 1, n)$. $\text{Rep}(Q, v)$ is a vector space, so we can pick a basis $\{\rho_p\}$. We identify $\text{Rep}(Q, v)^*$ with the opposite quiver via the trace form and let $\{\rho^p\}$ be the dual basis. Now, we can view $\rho^p, \rho_p$ as sitting inside $D \text{Rep}(Q, v)$. $\rho^p$ are coordinate functions on the vector space and $\rho_p$ can be seen as differentiation by $\rho^p$. Additionally, as $V$ is the representation space of the quiver, and hence also for the opposite quiver, both $\rho_p, \rho^p$ act on $V$. Thus, we can define the action of $Y_i, X_i$ via the operators:

$$Y_i \mapsto \sum_p \rho_p|_M \otimes (\rho^p)_i$$

$$X_i \mapsto \sum_p \rho^p|_M \otimes (\rho_p)_i$$

where $(\rho^p)_i$ means that the operator acts in the $i$th tensor factor. These operators are $g$-invariant and independent of the choice of basis and hence preserve $F_{n, \lambda}(M)$. All that remains is to define the $G(r, 1, n)$ action. The $S_n$ action is via permuting the tensor factors and $z_i$ acts by $z$ in the $i$th tensor factor where $z$ acts by $\zeta^i$ on $V_i$.

Before proceeding, we make some comments on $F_{n, \lambda}$.

Remark. (1) We can view $\chi_i$ as complex numbers by using $m \in \mathbb{C}$ to denote $\chi_i = m \frac{\mathbb{Z}}{\alpha_i}$.

(2) Note that the center of $G$ acts trivially on $M$ and acts by $n$ on $V^{\otimes n}$. Hence, for $F_{n, \lambda}$ to be nonzero, $\chi_1 + \cdots + \chi_r = n$.

(3) Additionally, since both $M$ and $V$ are actually equipped with an action of the group $G$ rather than just the Lie algebra $\mathfrak{g}$, $F_{n, \lambda}$ will be 0 unless each $\chi_i$ is an integral multiple of the trace. Using the convention of remark 1, this means that each $\chi_i$ must be an integer divisible by $\alpha_i$. 
2.9. **Parameter values for the image of** $F_{n,\chi}$. The last thing that we wish to explain in this section are the parameters $c, d$ for the rational Cherednik algebra whose modules $F_{n,\chi}$ maps to. In [Mon], Montarani relates the central character $\chi_i$ of $F_{n,\chi}$ with the parameters $\lambda$ and $\nu$ of the Gan-Ginzburg algebra:

$$\lambda_i = \frac{\chi_i}{\alpha_i} - \frac{1}{2}(2\alpha_i - \alpha_{i-1} - \alpha_{i+1})$$

and

$$\nu = -1,$$

where $\alpha = (\alpha_0, \ldots, \alpha_{r-1})$ is the dimension vector of the quiver. There is a slight discrepancy between our formulas and hers because the $\chi_i$ we use for the central character is $\alpha_i$ times her $\chi_i$. We want to translate these parameters to the parameters $t, c, d$ for the rational Cherednik algebra. To do so, we first introduce new parameters, $c_i$ for $i \in \mathbb{Z}/r\mathbb{Z}\{0\}$, which are related to the parameter $d$ by

$$c_l = \frac{1}{r} \sum_{j=0}^{r-1} \zeta^{-lj}(d_{j-1} - d_j).$$

To see this relation, note that the in the definition of the rational Cherednik algebra in [Mon], the main relation takes the form

$$[X_i, Y_i] = t - \frac{k}{2} \sum_{j \neq i} \sum_{l \in \mathbb{Z}/r\mathbb{Z}} s_{ij} z_{i}^l z_{i}^{-l} + \sum_{l \neq 0} c_i z_{i}^l.$$  

Hence, $c_0 = 0$ and

$$d_{j-1} - d_j = \sum_{l=0}^{r-1} \zeta^{lj} c_l.$$

We now use [GG][Theorem 3.5.2], which says that

$$\lambda_j = t + \sum_{l} c_l \zeta^{lj} = t + d_{j-1} - d_j$$

and

$$\nu = cr.$$

In principle, we can now compute the parameters for the Cherednik algebra that the functor $F_{n,\chi}$ maps to. Let us elaborate on these parameters in the equidimensional setting, when $\alpha_i = N$. In this case,

$$\lambda_i = \frac{\chi_i}{N}$$

Hence,

$$t = \frac{1}{r} \sum_i \lambda_i = \frac{n}{rN},$$

$$c = \frac{1}{r}$$

and

$$d_j - d_{j-1} = \frac{n - r\chi_j}{rN}.$$
3. Character Formulas from $F_{n,\chi}$.

Let $M$ be an equivariant $D$-module on $\text{Rep}(Q,\alpha)$. There is an action of $\mathbb{C}^*$ on $\text{Rep}(Q,\alpha)$ simply by scaling every representation. This action commutes with the $G$-action on $\text{Rep}Q$ and induces a $\mathbb{Z}$-grading on equivariant $D$-modules by eigenvalues of the Euler operator on $\text{Rep}(Q,\alpha)$. The Euler operator in $D(W)$ for any vector space $W$ is the operator

$$H = \sum_i x_i \frac{\partial}{\partial x_i}$$

where $x_i$ is the dual basis to some basis for $W$. To define character formulas for the $D$-modules we care about, we need to check that this operator has finite dimensional eigenspaces on the $D$-modules.

**Proposition 3.1.** If $M$ is any holonomic $D$-module on $\text{Rep}(Q,\alpha)$, then $H$ has finite dimensional eigenspaces on $M$.

**Proof.** By general theory of $D$-modules, it suffices to prove this fact for $i_*O_U$, where $U$ is any open subspace of $\text{Rep}(Q,\alpha)$. This module is generated by functions on $U$ by differential operators with constant coefficients, i.e., by the algebra of polynomials in $\frac{\partial}{\partial \rho}$. Thus, as a module over $\mathbb{C}[H]$, $i_*O_U$ has a surjective map from

$$\mathbb{C}[U] \otimes \mathbb{C}[\frac{\partial}{\partial \rho}]$$

Now, the right tensorand has finite dimensional $H$-eigenspaces and hence it suffices to prove that the left tensorand does. This follows from the fact that $\mathbb{C}[U]$ is a subset of a product of finitely many localizations of $\mathbb{C}[\text{Rep}(Q,\alpha)]$. \hfill \Box

If $M$ is now an equivariant holonomic $D$-module, then each graded piece is a $G$-representation, as $G$ commutes with the Euler operator. We can now define a character of $M$.

**Definition 3.2.** The character of $M$ for equivariant holonomic $M$ is

$$\text{ch}(M) := \sum_{i \in \mathbb{Z}} [M_i] z^i$$

where $[M_i]$ is the class of the degree $i$ piece of $M$ in the category of finite dimensional $G$-representations.

We can similarly grade the rational Cherednik algebra $H_{t,c,d}(n)$ by putting $X_i$ in degree $-1$, $Y_i$ in degree $1$ and $G(r,1,n)$ in degree $0$. We can grade the Verma modules and irreducibles in $O^-$ by simply specifying some degree for the lowest weight vector. In this case, such modules $M$ have graded pieces that are finite dimensional $G(r,1,n)$ representations.

**Definition 3.3.** The character of $M \in O_{\text{graded}}^-$ is

$$\text{ch}(M) := \sum_{i \in \mathbb{Z}} [M_i] z^i$$

where $[M_i]$ is the class of the degree $i$-piece of $M$ in the category of finite dimensional $G(r,1,n)$-representations.

Now, note that if $M$ is an equivariant $D$-module on the nilpotent cone, then $F_{n,\chi}(M) \in O^-$ inherits a grading from the grading on $M$ and this grading is compatible with the action of $H_{t,c,d}(n)$, since $X_i$ acts via degree 1 polynomials in the $D$-module part and $Y_i$ by degree 1 constant coefficient differential operators. In order to relate the character of $M$ with that of $F_{n,\chi}(M)$, we introduce the following definition.

**Definition 3.4.** Let $\lambda$ be some highest weight of $G$ (i.e. a nonincreasing sequence of $\alpha_i$-integers for each $GL(\alpha_i)$) and let $V(\lambda)$ be the associated representation. Then, the $\lambda$-character of a graded equivariant $D(\text{Rep}Q)$-module $M$
\[ \text{ch}_\lambda(M) = \sum_i m_{i,\lambda}(M) z^i \]

where \( m_{i,\lambda}(M) \) is the multiplicity of \( V(\lambda) \) in \( M_i \). Similarly, if \( M \) is a graded \( H_{t,c,d}(n) \)-module with finite dimensional graded pieces and \( \lambda \) is an \( r \)-partition, then

\[ \text{ch}_\lambda(M) = \sum_i c_{i,\lambda}(M) z^i \]

where \( c_{i,\lambda}(M) \) is the multiplicity of \( S(\lambda) \) in \( M_i \).

We can relate these two notions of character. The proof of the following proposition is an elementary exercise in Schur-Weyl duality.

**Proposition 3.5.** Suppose \( \lambda \) is a highest weight of \( G \). Let \( \lambda' \) be the dual highest weight and let \( \lambda' \otimes \chi \) be the highest weight of \( V(\lambda)^* \otimes \chi \). Then,

\[ m_{i,\lambda}(M) = \begin{cases} c_{i,\lambda'}(F_{n,\chi}(M)) & \text{if } \lambda' \otimes \chi \text{ is an } r \text{-partition of size } n \\ 0 & \text{else} \end{cases} \]

As a result of this proposition, we have

**Proposition 3.6.**

\[ [F_{n,\chi}(M)]_i = SW(M_i^* \otimes \chi) \]

where \( SW \) is the Schur-Weyl duality functor that sends \( V(\lambda) \) to \( S(\lambda) \) is \( \lambda \) is a highest weight that is an \( r \)-partition of \( n \) and kills all other \( V(\lambda) \).

An important consequence of these two propositions is that the character of \( M \) is determined by the characters of \( F_{n,\chi}(M) \) taken over all \( n \) and over all positive central characters \( \chi \). Thus, if we can compute the composition factors of the latter, then, in principle, we can obtain formulas for \( \text{ch}(M) \) using work of Rouquier [?].

### 4. \( F_{n,\chi}(i_*(O)) \) lies in Category \( \mathcal{O}^- \)

The main goal of this section is to prove that if \( O \) is a nilpotent orbit, then \( F_{n,\chi}(i_*(O)) \) lands in Category \( \mathcal{O}^- \). The first step is to get a more explicit description of the action of certain power sum symmetric polynomials in the \( X_i \) and \( Y_i \). The argument here is analogous to the argument in [CEE][Section 8.1-8.2]. We begin by describing the action of the rational Cherednik algebra on \( F_{n,\chi}(\mathbb{C}[\text{Rep}(Q,\alpha)]) \).

**Lemma 4.1.** View \( (\mathbb{C}[\text{Rep}(Q,\alpha)] \otimes V^{\otimes n})^g \) as \( (g,\chi) \)-equivariant maps \( \text{Rep}(Q,\alpha) \to V^{\otimes n} \). Then, for \( A \in \text{Rep}(Q,\alpha) \) and \( f \) a function as above,

\[ (X_i f)(A) = A_i(f(A)) \]

and

\[ (Y_i f)(A) = \sum \rho_i \partial \rho^i \]

where \( A_i \) is \( A \) acting on the \( i \)th coordinate of \( V^{\otimes n} \) and \( \{\rho^i\} \) is a basis for \( \text{Rep}(Q,\alpha)^* \) dual to \( \{\rho_p\} \).

**Proof.** The formula for the \( Y_i \) is by definition. The formula for the \( X_i \) comes from simplifying the formula

\[ \sum \rho^i(A) \otimes (\rho_p)_i. \]
Using this formula, we see that on \( F_{n,\chi}(\mathbb{C}[\text{Rep}(Q,\alpha)]) \), \( X_i^r \) acts as \( f(A) \mapsto A_i^r f(A) \). Hence, for some fixed value of \( m \), we have

\[
\sum_i (X_i^r m)(A) = \sum_i A_i^r m(f(A)).
\]

Since \( f \) is \((g,\chi)\)-equivariant, and \( A_i^r m \in g \), we get that

\[
\sum_i (X_i^r m)(A) = \chi(A_i^r m) f(A) - f([A_i^r m, A]) = \chi(A_i^r m) f(A).
\]

Thus, \( \sum_i X_i^r m \) acts on \( F_{n,\chi}(\mathbb{C}[\text{Rep}(Q,\alpha)]) \) as the function \( A \mapsto \chi(A_i^r m) \) on \( \text{Rep}(Q,\alpha) \). Similarly, \( \sum_i Y_i^r m \) must act as the Fourier transform of this function, which is some differential operator with constant coefficients, since \( Y_1 \) goes to \( X_1 \) under taking Fourier transform. We claim that this description holds true for any \( M \), not just for \( \mathbb{C}[\text{Rep}(Q,\alpha)] \).

**Corollary 4.2.** On \( F_{n,\chi}(M) \),

\[
\sum_i X_i^r m
\]

acts as the function \( A \mapsto \chi(A_i^r m) = \frac{n}{\chi} \text{Tr}(A_i^r m) \), and \( \sum_i Y_i^r m \) as its Fourier transform. Here, \( A_i^r m \) is an \( r \)-tuple of matrices from \( V_i \) to \( V_i \) for each \( i \), and \( A_0^r m \) is just the \( V_0 \) component.

**Proof.** By the Fourier transform use, it suffices to prove this for the \( X_i \). The second equality just uses the fact that \( \sum_i \chi_i = n \) and the fact that the trace of every matrix component of \( A_i^r m \) is the same. If we can prove that \( \sum_i X_i^r m \) acts purely on the \( D \)-module, then we will be done, because \( \mathbb{C}[\text{Rep}(Q,\alpha)] \) is a faithful \( D \)-module. To see this, note that

\[
\sum_i X_i^r m = \sum_i (\rho|_M \otimes (\rho|_i)^r m).
\]

Thus, the matrices that we get for each \( \rho|_M \) will be the same matrices acting on all \( i \). Again, these matrices are in \( g \), so we can use the \((g,\chi)\)-invariance to move them over to the \( D \)-module. Hence, this element acts only on the \( D \)-module and we are done.

We end with the main goal of this section.

**Proposition 4.3.** If \( M \) is supported on the nilpotent cone, then \( F_{n,\chi}(M) \) is in category \( O^- \).

**Proof.** Because \( H \) acts via finite dimensional eigenspaces on \( M \), \( F_{n,\chi}(M) \) is graded with finite graded pieces. Hence, it suffices to prove that \( \mathbb{C}[X_{i_1}, \ldots, X_{i_n}] \) acts locally nilpotently. Note that it suffice to prove that the algebra generated by \( \sum_i X_i^r m \) acts locally nilpotently, since the full polynomial algebra is finite over this algebra. But these elements act by traces of powers (since the \( \chi_i \) are just multiples of the trace at each vertex). These act locally nilpotently on \( M \) by the support condition, as the trace of a power of anything nilpotent must be 0 and hence such functions are in the ideal cutting out the nilpotent cone. This proves the first part of the definition of \( O^- \).

(\text{Need some work explaining how Euler element finiteness implies finite generation over } \mathbb{C}[Y].) \hfill \square

5. **Constructing Lowest Weight Vectors in** \( F_{n,\chi}(i_* O_O) \)

In this section, we assume \( \chi_i \geq 0 \) for each \( i \).

Since \( i_* O_O \) is supported on the closure of \( O \), if \( O \) is a nilpotent orbit, then Proposition 4.3 says that \( F_{n,\chi}(i_* O_O) \) lies in category \( O^- \). Hence, we would like to figure out its composition factors. To do so, we want to compute its lowest weight vectors. In this section, we construct a space of lowest weight vectors in \( i_* O_O \). In subsequent sections, we will use this space to characterize \( F_{n,\chi}(\text{IC}(O)) \) under certain assumptions and then use this characterization to obtain character formulas.
The construction works for all $r$ but we will give the construction for $r = 2$ first and then explain how to generalize, since it is easier to understand what is going on this way, primarily due to less cumbersome notation. This construction is a direct generalization of Lemmas 9.13 to 9.15 of [CEE].

First, recall that $\mathbb{C}[O] \subseteq i_* \mathcal{O}_O$. So, we can construct a lowest weight vector in

$$(\mathbb{C}[O] \otimes V \otimes^n)_{\chi}^g.$$  

We view this space as $(g, \chi)$-equivariant functions from $O$ to $V \otimes^n$. In section 2.2, we associated a 2-partition $\lambda(O)$ to the orbit $O$. Suppose this 2-partition is $(\lambda^1, \lambda^2)$.

We first construct equivariant functions when $\chi_0 = \alpha_0, \chi_1 = \alpha_1$. Since $\chi_i$ is always a multiple of $\alpha_i$, these should be viewed as the atomic functions, and all other functions giving lowest weight vectors will be constructed out of these.

**Case 1:** Suppose $\chi_0 = 0, \chi_1 = 0$. Remember that this also implies that $n = \alpha_0$. Let $\lambda^0 = (\mu_1, \ldots, \mu_s)$ and $\lambda^1 = (\omega_1, \ldots, \omega_s')$. Suppose $X$ is in $O$. Starting with $s$ vectors $v_i$ in $V^\ast_0$, we can construct vectors

$$(X^*)^j(v_i)$$

for $j$ even from 0 to $\mu_i - 1$. By the evenness condition, these are all elements in $V^\ast_0$. Additionally, starting with $s'$ vectors $w_i$ in $V^\ast_1$, we can construct vectors

$$(X^*)^j(w_i)$$

for $j$ odd from 1 to $\omega_i - 1$. By the parity condition, these are also elements in $V^\ast_1$ and by the way in which we defined $\lambda(O)$, the total amount of vectors that we get is precisely $\alpha_0$. Hence, we can take the determinant of the matrix formed by all these vectors (after specifying some order). Compactly, omitting the transpose, we can denote this by

$$f_0(X, v_1, \ldots, v_s, w_1, \ldots, w_{s'}) = \left( \bigwedge_{i}^{\mu_i - 1} X^j(v_i) \right) \left( \bigwedge_{i}^{\omega_i - 1} X^j(w_i) \right).$$

This is a $\mathbb{C}$-valued homogeneous function on $O \times (V^\ast_0)^s \times (V^\ast_1)^{s'}$ that is $\chi$-equivariant (since $\chi_0$ is just the trace and the function changes as the determinant.) Looking at the multidegree of the function, we see that

$$f_0 \in \left( \bigotimes_{i}^{m_i - 1} S_{\lambda^0_i}^i(V_0) \bigotimes_{i}^{m_i - 1} S_{\lambda^1_i}^i(V_1) \right)^{g, \chi} \subseteq (i_* \mathcal{O}_O \otimes V \otimes^\alpha)_{\chi}^g.$$  

**Case 2:** In case 2, we have $\chi_0 = 0, \chi_1 = \alpha_1$ and we define $f_1$ in a completely analogous manner, just replacing the roles of $V^\ast_0, V^\ast_1$. We thus get

$$f_1 \in (i_* \mathcal{O}_O \otimes V \otimes_{\alpha_1})_{\chi}^g.$$  

**General Case:** Suppose $\chi_i \geq 0$ divisible by $\alpha_i$ (which is necessary for $F_{n, \chi}$ to be nonzero.) Then, we find

$$f := f_0^{\lambda_0} f_1^{\lambda_1} \in (i_* \mathcal{O}_O \otimes V \otimes^n)_{\chi}^g.$$  

The power of $f_i$ can be described in two ways. Viewing it as a tensor, it can be constructed by multiplying in the $\mathbb{C}[O]$ part and appending tensors. A second viewpoint can be obtained if we look at it as a homogeneous function on

$$O \times (V^\ast_0)^s \times (V^\ast_1)^{s'},$$

where it is just the honest power of a function. The $\chi$-equivariance is automatic from the equivariance of each $f_i$.

We now pick a very special element in $O$ and look at where $f_0$ and $f_1$ send this element.
**Definition 5.1.** Take the bipartition \( \lambda(O) = (\lambda^0, \lambda^1) \). Choose a basis \( \rho_1, \ldots, \rho_{\alpha_0} \) for \( V_0^* \) and \( \xi_1, \ldots, \xi_{\alpha_1} \) for \( V_1^* \). Fill in the basis vectors into the bi-partitions as follows: when filling in the basis vectors for \( V_0^* \), fill first in the even columns for \( \lambda^0 \), then the odd columns for \( \lambda^1 \). When filling in the basis vectors for \( V_1^* \), follow the above rule but with \( \lambda^0 \) and \( \lambda^1 \) swapped. Now, define the element \( J \in \text{Rep}(Q, \alpha) \) as the matrix whose transpose is the map obtained by moving right along the basis vectors filled into \( \lambda(O) \).

It is clear from the definition of \( \lambda(O) \) that \( J \) is in \( O \). We compute \( f_0 \) and \( f_1 \) evaluated at \( J \).

**Lemma 5.2.** Let \( \mu^* = \lambda^1*, \omega^* = \lambda^2* \) be the conjugate partitions (the transpose in terms of Young diagrams). Recall that \( s = \mu_1^*, s' = \omega_1^* \) as these were the number of parts of \( \lambda^1 \) and \( \lambda^2 \) respectively. Hence we may define for any \( \mu \), \( \omega \)

\[
\Delta_{\mu^*}(v_1, \ldots, v_s)
\]

as the top left \( \mu^*_i \) minor in the \( \rho \)-basis and similarly for \( \omega^*_i \) and the \( w_j \) in the \( \xi \)-basis. Here \( \mu^*_i \) is the \( i \)th row of \( \mu^* \), equivalently the \( i \)th column of \( \lambda^1 \), where our convention is that we start the numbering of rows and columns with 0, not 1. When we take minors, we put the vectors into the columns of the matrix. With this notation, we have (up to sign)

\[
f_0(J, v_1, \ldots, v_s, w_1, \ldots, w_{s'}) = \prod_{i \text{ even}} \Delta_{\mu^*_i}(v_1, \ldots, v_s) \prod_{i \text{ odd}} \Delta_{\omega^*_i}(w_1, \ldots, w_s)
\]

and

\[
f_1(J, v_1, \ldots, v_s, w_1, \ldots, w_{s'}) = \prod_{i \text{ odd}} \Delta_{\mu^*_i}(v_1, \ldots, v_s) \prod_{i \text{ even}} \Delta_{\omega^*_i}(w_1, \ldots, w_s).
\]

**Proof.** We do the computation for \( f_0 \) since the \( f_1 \) case is completely analogous. Let us look at which vectors we get from \( v_i \). These are the even powers of \( J \) applied to \( v_i \). So we really need to understand how \( J^2 \) acts on \( v_i \). We break this down into steps:

1. Since \( \rho_1, \ldots, \rho_{\mu_0}^* \) are not in the image of \( J^2 \), the first \( \mu_0^* \) entries of \( J^2 v_i \) are 0.
2. Similarly, in \( J^4 v_i \), the first \( \mu_0^* + \mu_2^* \) entries are 0.
3. For \( J^{2k} v_i \), the first \( \mu_0^* + \cdots + \mu_{2k}^* \) entries are 0.

Let us now look at what we get from \( J^{2k+1} w_i \). Essentially by the same argument as before, the first \( \omega_1^* + \cdots + \omega_{2k+1}^* \) terms of this vector are 0. Additionally, the terms coming from the \( v_i \) don’t interact with the terms coming from the \( w_i \).

The upshot of all this is the following. Suppose we reorder the vectors in the wedge product so that we first put all the \( v_i \), then the \( J^2 v_i \), then the \( J^4 v_i \), and so on. Next, we put the \( J w_i \) and then the \( J^3 w_i \), and so on. This reordering may pick up a minus sign but won’t change the determinant otherwise. If we do this, then the matrix whose determinant gives \( f_0 \) is

1. Block diagonal between the \( v_i \) terms and the \( w_i \) terms.
2. Within each \( v_i \) segment or \( w_i \) segment, it is block upper triangular.
3. The determinant of the first \( v_i \) matrix in \( \Delta_{\mu_0^*}(v) \). The determinant of the second \( v_i \) matrix is \( \Delta_{\mu_2^*} \) and so on.
4. The determinant of the first \( w_i \) matrix is \( \Delta_{\omega_1^*}(w) \) and so on.

This gives us the formula we stated for \( f_0 \).

\[ \Box \]

The image of \( J \) under \( f_0, f_1 \) or \( f \) is thus some very explicit element of \( V^{\otimes n} \). We claim that this is a highest weight vector for the \( G \)-action and compute its weight, which will be some \( r \)-partition. For \( j \in \mathbb{Z}/r\mathbb{Z} \), let \( \mu^j, \omega^j \) be the partitions obtained by only taking those columns of \( \mu, \omega \) whose index is equal to \( j \) mod \( r \).

**Proposition 5.3.** Recall the definition of \( f \)

\[ f = \frac{x_n}{f_0^{x_0}} \frac{x_1}{f_1^{x_1}}. \]

\( f(J) \in V^{\otimes n} \) is a highest weight vector for the \( G \)-action with weight \( (\beta^0, \beta^1) \)
\[ \beta_0^i = \frac{\lambda_0}{\alpha_0} \left( \frac{\mu_1}{2} \right) + \frac{\lambda_1}{\alpha_1} \left( \frac{\mu_1 - 1}{2} \right) = \frac{\lambda_0}{\alpha_0} \mu_0^i + \frac{\lambda_1}{\alpha_1} \mu^1 \]

and

\[ \beta_1^i = \frac{\lambda_1}{\alpha_1} \left( \frac{\omega_1}{2} \right) + \frac{\lambda_0}{\alpha_0} \left( \frac{\omega_1 - 1}{2} \right) = \frac{\lambda_1}{\alpha_1} \omega_0^i + \frac{\lambda_0}{\alpha_0} \omega^1. \]

In other words, we obtain \( \beta^0 \) by taking a weighted sum of \( \mu^0 \) and \( \mu^1 \) and we get \( \beta^1 \) by taking a weighted sum of \( \omega^0 \) and \( \omega^1 \).

**Proof.** Since a tensor product of highest weight vectors is a highest weight vector of the sum of the weights, it suffices to prove this fact for \( f = f_0, f_1 \) and hence by the description of \( f \) in terms of appending of tensors. We will actually only check this for \( f_0 \) since the other case is analogous.

We need to prove that strictly upper triangular matrices in \( g \) kill \( f(J) \) and diagonal matrices multiply by the weight above. The fact that strictly upper triangular matrices kill \( f(J) \) follows from two facts. First, if \( A \) is strictly upper triangular, then \( Af(J) = f([A, J]) \) because \( \chi(A) = 0 \) and because \( f \) is \( \chi \)-invariant. But \([A, J]\) is on the boundary of \( O \) and on the boundary, the function is uniformly 0, as one of the \( X^k \) involved in the determinantal definition becomes the 0-matrix. Basically, the idea is that on the boundary, we have to refine the partition somehow by breaking it up.

We then need to check that \( f \) is an eigenvector for the diagonal matrices and compute the eigenvalue. The fact that it is an eigenvector is obvious from the previous proposition. So, we just compute the eigenvalue. The only case we need to check is then \( f = f_0 \) for which \( \chi_0 = \alpha_0, \chi_1 = 0 \). So, we need to check that

\[ \beta_0^0 = \left[ \frac{\mu_1}{2} \right] \]

and

\[ \beta_1^0 = \left[ \frac{\omega_1 - 1}{2} \right]. \]

Again, we will only check that \( \beta_0^0 \) is as stated since the other case is analogous. We use the description of \( f_0 \) furnished by the above lemma. The weight of the minor

\[ \Delta_j(v_1, \ldots, v_s) \]

is \((j, \ldots, j, 0, \ldots, 0)\) where we have \( j \) many \( j \)'s. Hence, \( \beta_0^0 \), which is the eigenvalue of the matrix \( E_{i+1,i+1} \in \mathfrak{gl}(V) \), is the number of \( \mu_j^* \) with \( j \) even such that \( \mu_j^* \leq i + 1 \). This is precisely the number of even column terms in \( \mu_i \), which is

\[ \left[ \frac{\mu_1}{2} \right], \]

as desired.

As a result of the above proposition and Schur-Weyl duality, we see that the \( G(r, 1, n) \) representation generated by \( f \) is the irreducible representation corresponding to

\[ \left( \frac{\lambda_0}{\alpha_0} \mu_0^0 + \frac{\lambda_1}{\alpha_1} \mu_1^1, \frac{\lambda_0}{\alpha_0} \omega_0^1 + \frac{\lambda_1}{\alpha_1} \omega^0 \right). \]

**Proposition 5.4.** \( f \) is a lowest weight vector in \( F_{n,\chi}(i_*O_O) \) of the above \( G(r, 1, n) \)-weight.

**Proof.** All we need to show is that the \( X_i \) kill \( f \). Since \( f \) is \((g, \chi)\)-equivariant and \( O \) is a single orbit, it suffices to show that \( X_i \) kill \( f(J) \). Now, the formula for the action of \( X_i \) is given by the formula in Lemma 4.1, since the action of \( X_i \) on \( \mathbb{C}[O] \subseteq i_*O_O \) comes from the action on \( \mathbb{C}[\text{Rep}(Q, \alpha)] \). Hence, we just need to see that multiplying \( f(J) \) in any single tensor factor is 0. But multiplying by \( J \) kills the first \( \mu_i^0 \) terms of the \( v_i \) and kills the first \( \omega_0^0 \) terms of the \( w_i \). Thus, all relevant minors vanish and we are done.

\[ \square \]
We now explain how to go from the case \( r = 2 \) to the general case. The way we found \( f_0 \) and \( f_1 \) is to take the determinant function given in [CEE][Lemma 9.13] and realize that we can break this determinant up into the terms that come from \( V_0 \) and those that come from \( V_1 \). We can now do the same for \( r > 2 \), i.e., breaking up the terms into those that come from each \( V_i \) to get \( r \) many functions \( f_0, \ldots, f_{r-1} \). We can then replicate the Lemma and both Propositions above with very little change, with the only tricky thing to remember that the transpose of any representation of the quiver shifts degrees by \(-1\). The upshot is that for any \( \chi \) with \( \chi_i \geq 0 \) and divisible by \( \alpha_i \), we get an element \( f \in F_{n,\chi}(i_*O_O) \) that comes from \( C[i_*O_O] \) and satisfies the following theorem.

**Theorem 5.5.** Let \( \lambda(O) = (\lambda^1, \ldots, \lambda^r) \) be the \( r \)-partition corresponding to \( O \). Let \( \lambda^{k,i} \) be the \( i \mod r \) part of \( \lambda^k \). The element \( f \in F_{n,\lambda}(i_*O_O) \) generates under \( G(r,1,n) \) a lowest weight space for \( H_{l,c,d}(n) \) with \( G(r,1,n) \)-weight the irreducible representation corresponding to the \( r \)-partition \( \mu = \mu(O) = (\mu^1, \ldots, \mu^r) \) where

\[
\mu^k = \sum_{j=0}^{r-1} \frac{\chi_{k-j}}{\alpha_{k-j}} \lambda^{k,j}.
\]

In particular, this means that

\[
\mu_i^k = \sum_{j=0}^{r-1} \frac{\chi_{k-j}}{\alpha_{k-j}} \left\lceil \frac{\lambda^k_j - j}{r} \right\rceil.
\]

**Definition 5.6.** For \( O \) a nilpotent orbit, let \( L(\mu(O)) \) denote the irreducible representation of \( H_{l,c,d}(n) \) with lowest weight \( \mu(O) \).

The following corollary of the theorem will be very useful in showing that in many cases, this lowest weight vector lives in and characterizes \( F_{n,\chi}(IC(O)) \).

**Corollary 5.7.** Suppose \( \chi_i > 0 \) for all \( i \). Then, if \( O \) and \( O' \) are different orbits, then \( \mu(O) \) and \( \mu(O') \) are distinct.

**Proof.** The ceiling function is monotonically increasing and if we look at the combination of all functions

\[
\left\lceil \frac{x - j}{r} \right\rceil
\]

then one of these will definitely increase at each integral value of \( x \). Hence, if \( \lambda(O) \) and \( \lambda(O') \) are distinct, then so are \( \mu(O), \mu(O') \). The result then follows from the fact that \( \lambda(O) \) characterizes \( \mu(O) \).

\( \square \)

### 6. Irreducibility Conjecture

In this section, we will state our conjecture regarding the irreducibility of \( F_{n,\chi}(IC(O)) \) and use it to compute its highest weight. This conjecture holds outside a finite range of central characters for \( r = 2 \). However, it is not clear as of now whether this conjecture holds for positive central characters or for \( r > 2 \). There is, however, some evidence for this conjecture that we will discuss in subsequent sections.

**Irreducibility Conjecture:** \( F_{n,\chi} \) sends irreducible equivariant \( D \)-modules supported on the nilpotent cone to irreducible objects in category \( \mathcal{O}^- \) for \( H_{l,c,d}(n) \).

Note that the irreducible equivariant \( D \)-modules supported on the nilpotent cone are precisely \( IC(O) \) for nilpotent orbits \( O \). The main consequence of this conjecture is the following important theorem.

**Theorem 6.1.** Suppose \( \chi \) is a central character with \( \chi_i > 0 \) for all \( i \). Then, under the assumption that the irreducibility conjecture holds,

\[
F_{n,\chi}(IC(O)) = L(\mu(O)).
\]
Proof. We prove this theorem by Noetherian induction on the support $\mathcal{O}$ of $IC(O)$. The base of the induction are the closed orbits $O$. For closed orbits, $IC(O) = i_*\mathcal{O}_O = \mathcal{O}_O$. Now, by Theorem 5.5, $L(\mu(O))$ is a composition factor of $F_{n,\chi}(i_*\mathcal{O}_O)$ Hence, by the irreducibility conjecture, it must be all of $F_{n,\chi}(IC(O))$.

Let $O$ now be any orbit and suppose the statement holds for all orbits supported on the boundary of $O$. $i_*\mathcal{O}_O$ has as composition factors, $IC(O)$ with multiplicity 1 and $IC(O')$ with unknown multiplicity $m_{O,O'}$ for $O'$ orbits in the boundary of $O$. Hence, as $F_{n,\chi}$ is an exact functor,

$$[F_{n,\chi}(i_*\mathcal{O}_O)] = [F_{n,\chi}(IC(O))] + m_{O,O'}[F_{n,\chi}(IC(O'))]$$

by the induction hypothesis. Now, $L(\mu(O))$ is a composition factor of $F_{n,\chi}(i_*\mathcal{O}_O)$ by Theorem 5.5. Also, by Corollary 5.7, $L(\mu(O)) \neq L(\mu(O'))$ for any $O'$ in the boundary of $O$. Thus, $L(\mu(O))$ must be a composition factor of $F_{n,\chi}(IC(O))$ and hence must be its only composition factor by the irreducibility conjecture.

\[ \square \]

7. Evidence for the Conjecture

For the rest of this paper, we will show where we know the conjecture holds and discuss evidence for the conjecture in general. For this section, we assume that we are in the equidimensional case $\alpha_i = N$. 

7.1. Sphericalization: definition. Inside $H_{t,c,d}(n)$, we can find an idempotent $e = \frac{1}{|G(r,1,n)|} \sum_{g \in G(r,1,n)} g$.

**Definition 7.1.** The spherical rational Cherednik algebra $SH_{t,c,d}(n)$ is $eH_{t,c,d}(n)e$.

Given a left $H_{t,c,d}(n)$ module $M$, $eM$ is a module over $SH_{t,c,d}(n)$.

**Definition 7.2.** The functor $M \mapsto eM$ is known as the sphericalization functor.

By standard facts regarding associative algebras, sphericalization is an equivalence of categories if and only if the two sided ideal generated by $e$ is the identity.

7.2. Quantum Hamiltonian Reduction. In [Obl] and [Gor], it is shown that the spherical Cherednik algebra $SH_{t,c,d}(N)$ is isomorphic to the quantum Hamiltonian reduction of $D(\text{Rep} Q, N)$. In this section, we recall both the general construction of quantum Hamiltonian reduction and the specific case of $D(\text{Rep} Q)$ studied by Oblomkov and Gordon.

Recall that the group $G$ acts on $\text{Rep}(Q, N)$ by automorphisms and hence also acts on $D(\text{Rep} Q, N)$. Since $V_0$ is a $G$-representation, $G$ also acts on

$$D(\text{Rep} Q, N) \boxtimes D(\mathbb{P}(V_0), \mathcal{O}(n)) = D(\text{Rep}(Q, N) \times \mathbb{P}(V_0), \mathcal{O}(n))$$

where $D(\mathbb{P}(V), \mathcal{O}(n))$ is the algebra of global differential operators on $\mathbb{P}(V)$ twisted by the line bundle $\mathcal{O}(n)$. Hence, we get a locally finite $\mathfrak{g}$-module structure on this algebra. We denote the algebra by $D$ and for each $X \in \mathfrak{g}$, the element $X_D$ as the operator on $D$.

The action of $G$ on $\text{Rep}(Q, N) \times \mathbb{P}(V_0)$ along with the fact that $\mathcal{O}(n)$ is a $G$-equivariant line bundle also gives us a homomorphism $\phi: \mathfrak{g} \to D$ which is a quantum comoment map, i.e., it satisfies

$$[\phi(X), -] = X_D.$$

**Definition 7.3.** Given a quantum comoment map $\phi$, and character $\chi$, let $I_\chi$ be the left ideal generated by $\phi(X) - \chi(X)$ for each $X \in \mathfrak{g}$. The quantum Hamiltonian reduction of $D$ is

$$D_\chi := (D/I_\chi)^\theta.$$

Although $I_\chi$ is a left ideal only, it is easy to see that the multiplication in $D$ induces an algebra structure on $D_\chi$. Additionally, since $G$ is reductive,

$$D_\chi = D^\theta/I_\chi^\theta.$$
An important property of quantum Hamiltonian reduction is that we not only get an algebra out of the construction but also a functor from modules over $D$ to modules over $D_X$.

**Definition 7.4.** The functor of quantum Hamiltonian reduction, denoted $QHR$, from modules over $D$ to modules over $D_X$ sends $M$ to $M^\oplus$ and morphisms to their restrictions.

The key property of this functor that we use is that it sends irreducibles to irreducibles.

In [Gor], using the construction of a radial part homomorphism in [Obl], it is shown that the quantum Hamiltonian reduction $D_X$ of the algebra $D$ above is isomorphic to the spherical part of the rational Cherednik algebra $SH_{t,c,d}(n)$. The parameter values $t, c, d$ are determined in [Obl] and [Gor] as follows:

1. $t = 1$.
2. $c = \frac{2}{N}$
3. $d_{j - 1} - d_j = 1 - \frac{rN}{N}$ for $j > 1$.

This isomorphism is established by a radial part homomorphism constructed by Oblomkov in [Obl][Section 2.1] from $D(Rep Q, N)\oplus$ to what turns out to be the image of $SH_{t,c,d}(n)$ in $D(h^{\mathfrak{rep}})$ under the Dunkl embedding. Here, $h^{\mathfrak{rep}}$ is the open subspace of the reflection representation of $G(r, 1, N)$ on which the action of $G(r, 1, N)$ is free. Written in the basis $Y_i$, a description of this space is that the ratio of any two coordinates cannot be an $r$th root of unity and no coordinate can be 0. We will not describe this homomorphism in full and just use what we need in later sections.

### 7.3. Comparing $eF_{n,\chi}$ to QHR

In this section, we assume $r = 2$. Our goal is to show that the action of $SH_{t,c,d}(n)$ on $eF_{n,\chi}(C[Rep(Q, N)])$ gives a surjective homomorphism

$$\rho : SH_{t,c,d}(n) \to SH_{t',c',d'}(N)$$

where the latter spherical Cherednik algebra comes from quantum Hamiltonian reduction. Furthermore, we wish to show that for any $D$-module $M$

$$eF_{n,\chi}(M) = \rho^*(QHR(M \otimes S^nV_0)).$$

Here, note that $S^nV_0$ is the global sections of $\mathcal{O}(n)$ on $V_0$ and hence $M \otimes S^nV_0$ is an irreducible module over $n$-twisted $D$-modules on $(Rep(Q, N) \times \mathbb{P}(V_0))$ and thus its quantum Hamiltonian reduction is defined.

Here is how $\rho$ is constructed. First, we note that the action of $SH_{t,c,d}(n)$ on $eF_{n,\chi}(C[Rep(Q, N)])$ gives a map from $SH_{t,c,d}(n)$ to

$$D = D(Rep Q, N) \boxtimes D(\mathbb{P}(V_0), \mathcal{O}(n)).$$

The map $\rho$ is the composition of this map with the radial part homomorphism of [Obl] and [Gor].

**Proposition 7.5.** For $r = 2$, the map $\rho$ is surjective.

**Proof.** For $r = 2$, it is known that $SH_{t,c,d}(N)$ is generated by $C[\mathfrak{h}]^{G(r,1,N)}$ and $C[\mathfrak{h}^*]^{G(r,1,N)}$ because they Poisson generate the associated graded algebra. (Much more needed here).

Hence, it suffices to show that $\rho$ hits the $G(r, 1, N)$ symmetric functions in $X_i$ and $Y_i'$, where $Y_i'$ gives a basis for $\mathfrak{h}$ and $X_1', \ldots, X_n'$ the dual basis for $\mathfrak{h}^*$. These are generated by the power sum symmetric functions

$$\sum_i (X'_i)^2, \sum_i (Y'_i)^2$$

so we just need to show that these lie in the image of $\rho$. Now,

$$eF_{n,\chi}(C[Rep(Q, N)]) = (C[Rep Q] \otimes S^n(V_0))^\chi$$

is the quantum Hamiltonian reduction of $C[Rep Q] \otimes S^n(V_0)$. As an $H_{t',c',d'}(N)$-module, we identify it with $C[\mathfrak{h}]^{G(2,1,N)}$ as follows:

We identify $\mathfrak{h}$ inside $Rep Q$ by first identifying $\mathfrak{h}$ with diagonal $N \times N$ matrices and then mapping $\mathfrak{h}$ into $Rep Q$ by taking the same diagonal matrix for each arrow. If we view

$$(C[Rep Q] \otimes S^n(V_0))^\chi$$
as \( g \), \( \chi \)-equivariant functions from \( \text{Rep}(Q, N) \) to \( S^n(V_0) \), then the restriction to \( \mathfrak{h} \) inside \( \text{Rep} Q \) determines the function as \( \mathfrak{h} \) is a slice for the \( G \)-action on \( \text{Rep} Q \) (see [Obl][2.1]). Additionally, \( \mathfrak{h} \subseteq \text{Rep}(Q, N) \) is acted on trivially by the Lie algebra \( \mathfrak{g} \subseteq g \) where we take the same diagonal matrices at each vertex. By equivariance, any function in this space must send \( \mathfrak{h} \) to the weight 0-space in \( S^n(V_0) \) and can be viewed as an element of \( \mathbb{C}[\mathfrak{h}]^{G(2,1,n)} \). Conversely any element of the latter space that is a multiple of the discriminant can be obtained this way. Thus, restricting to \( \mathfrak{h} \), we can identify

\[
e_{F_{n,\chi}}(M)
\]

with \( \mathbb{C}[\mathfrak{h}]^{G(2,1,n)} \), and this identification is compatible with quantum Hamiltonian reduction, and hence describes \( e_{F_{n,\chi}}(M) \) as a module over \( H^*_{t,c,d}(N) \).

Let us now use the formulas of Corollary 4.2 to compute formulas for the action of \( \sum_i X_i^{2k} \subseteq \text{SH}_{t,c,d}(n) \) on \( \mathbb{C}[\mathfrak{h}]^{G(r,1,N)} \) (and similarly for the \( Y \)'s). For \( A \in \mathfrak{h} \),

\[
\sum_i X_i^{2k}f(A) = \chi(A^{2k})f(A)
\]

But \( A \) is identified inside \( \mathfrak{g} \) by the same diagonal matrix everywhere so

\[
\chi(A^{rk}) = \sum_i \chi_i(A^{2k}) = \frac{n}{N} \text{Tr}(A^{2k})
\]

and hence \( \sum_i X_i^{rk} \) acts as multiplication by \( \sum_i (X_i')^{2k} \). Thus, \( \rho \) hits all power sum symmetric functions in \( (X_i')^2 \).

For the computation with the power symmetric \( Y \)-operators, let us begin with the example of where \( \rho \) sends

\[
\sum_i Y_i^2 \in \text{SH}_{t,c,d}(n).
\]

Let us use \( E_{ij}^0, E_{ij}^1 \) as bases for \( \text{Rep}(Q, N) \) where the superscript indicates the domain vector space of the map. Using the fact that \( \sum_i Y_i^2 \) acts via the Fourier transform of the function by which \( \sum_i X_i^2 \) acts, we get that

\[
\sum_i Y_i^2f(A) = \frac{n}{N} \sum_{i,j,k=0,1} \frac{\partial}{\partial E_{ij}^k} \frac{\partial}{\partial E_{ji}^{k+1}}.
\]

Thus, \( \rho \) sends \( \sum_i Y_i^2 \) to the radial part of the operator on the right hand side, which is

\[
\sum_i \frac{\partial}{\partial X_i^2}.
\]

But this is precisely how \( \sum_i (Y_i')^2 \) acts in the spherical polynomial representation (since the remaining part of the Dunkl operators are killed by \( e \)). The proof for the other power sums is similar.

(THE MIGHT NEED MAJOR FIXING)

\[\square\]

**Corollary 7.6.** For any \( M \),

\[\rho^* \circ QHR = e_{F_{n,\chi}}.\]

**Proof.** This fact follows from corollary 4.2, as the same formulas now apply to all \( M \). Here, we also use the fact that the spherical polynomial representation, which we studied in the proposition above, is faithful, and hence the quantum Hamiltonian reduction map can also be read off from the action on the spherical polynomial representation.

\[\square\]

**Corollary 7.7.** If \( M \) is irreducible, then so is \( e_{F_{n,\chi}}(M) \).
Proof. This follows from the fact that quantum Hamiltonian reduction sends irreducibles to irreducibles and the fact that $\rho^* \circ QHR = eF_{n,\chi}$.

\[\square\]

7.4. Equivalence of Sphericalizing. In this section, we don’t assume $r = 2$ but we still assume that we are in the equidimensional setting.

Definition 7.8. We say that the parameter $t, c, d$ is spherical if the sphericalization functor is an equivalence of categories.

The parameter being spherical is equivalent to the two sided ideal generated by $e$ being everything at that parameter value. In [CEE], the parameter was spherical and this, combined with analysis similar to that in the previous section, sufficed to prove the irreducibility conjecture. Unfortunately, we are not always at spherical parameter in our setting. Aspherical parameters have been classified in [DG].

Proposition 7.9. [DG][Theorem 1.1] The parameter $(t, c, d)$ is aspherical if and only if one of the following two hold:

(a) $c = \frac{-k}{m}$ for integers $k$ and $m$ satisfying $1 \leq k < m \leq n$.

(b) There exists an integer $0 \leq l \leq r - 1$, an integer $-n + 1 \leq m \leq n - 1$, an integer $1 \leq k \leq r - 1$ and a sufficiently small $\alpha > 0$ such that

$$k + \alpha r = \frac{d_l - d_{l-k}}{t} + \frac{rmc}{t}.$$

We don’t go into the specifics about $\alpha$ being sufficiently small but the reader can look the bound up in [DG].

The first property never holds since $c > 0$ for us. But the second property will often hold. Let us look at what happens when we are in the equidimensional setting when $k < r$. We have

$$\frac{d_l - d_{l-k}}{t} = k - \frac{r}{n} \sum_{j=l}^{l-k+1} \chi_j$$

and

$$c = \frac{N}{t}.$$

Hence, property (b) is equivalent to to the existence of $-n + 1 \leq m \leq n - 1$ some $l, k$ and some sufficiently small $\alpha$ such that

$$na = mN - \sum_{j=l}^{l-k+1} \chi_j.$$

This will obviously hold if $\chi_j$ are all bigger than 0, as in that case, each $\chi_j$ is also smaller than $n$ and we can just choose $\alpha$ to be 0. This property will fail to hold for central characters where the sums $\sum_{j=l}^{l-k+1} \chi_j$ become sufficiently large in absolute value for each $l$ and $k$, as $\alpha$ and $m$ are bounded.

References


