AFFINE HECKE ALGEBRAS OF TYPE A AND THEIR CYCLOMATIC QUOTIENTS

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Abstract. These are notes for a talk in the MIT-NEU Graduate seminar on Hecke Algebras and Affine Hecke Algebras (AHAHA) held in Fall 2014. This talk is divided into two parts. The first deals with Ariki-Koike algebras, also called Cyclotomic Hecke Algebras, which are Hecke algebras associated to the infinite family of complex reflection groups of type $G(m, 1, n)$. We describe two common bases for this family of algebras, one in terms of Jucys-Murphy elements and the other in terms of reduced words in the complex reflection groups, and then construct a linear form which symmetrizes the algebra.

The second part of the talk is related to Affine Hecke Algebras, for which we construct a basis, give an explicit description of the center, and then prove Kato’s Theorem regarding irreducibility of certain induced representations known as Kato Modules. We end the talk by formulating and proving the Chang-Rouquier equivalence. The main references for this talk are [GJ11, 5.1-5.2], [Kle05, 3.1-3.4, 4.1-4.3] and [CR04, 3.1-3.2]. Auxilliary useful references are [AK94, BM97, MM98] and [Mac95, 1, Appendix B]

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Part A - Affine Hecke Algebra of Type A

1. Introduction and Notation

The Affine Hecke algebra (of Type $A_n$) is defined as a $q$-deformation of the group algebra of the affine Weyl group (of Type $A_n$). In this section of the talk, our goal is to construct a basis for the affine hecke Algebra, describe the center of the algebra and then give the construction and proof of irreducibility of Kato modules, which are modules induced from the Laurent polynomial subalgebra of the affine Hecke
algebra. At the end of the section, we prove the equivalence of categories given in [CR04]. We will only
discuss the nondegenerate affine Hecke algebra. There are analogous results with similar proofs in the
degenerate case which we will leave as an exercise (and which can be looked up in [Kle05, 3, 4].)

We fix some notation.

Notations and Conventions
(1) $F$ denotes a commutative domain (most often a field), not necessarily of characteristic 0.
(2) For $q$ a generic variable, $A$ is the commutative ring $F[q^\pm]$ and $K$ is its field of fractions.
(3) $\mathcal{H}_{F,n}$ (and later $\mathcal{H}_n$) denotes the affine Hecke algebra over $F$.
$\mathcal{H}_{A,n}$ denotes the affine Hecke algebra over $A$ with parameter equal to the polynomial variable $q$.
(4) We let $\mathcal{P}_{F,n}, \mathcal{P}_{A,n}, \mathcal{P}_n$ be the respective Laurent polynomial subalgebras and let
$Z_{F,n}, Z_{A,n}, Z_n$ be the symmetric Laurent polynomials, which will turn out to be the
centers of the affine Hecke algebras.
(5) We define $\mathcal{H}_{F,n}^T, \mathcal{H}_{A,n}^T, \mathcal{H}_n^T$ to be the (regular) Hecke subalgebra of the affine Hecke
algebra.
(6) For $s \in S_n$, and $f \in \mathcal{P}_{F,n}$ we define $s \cdot f$ to be $f$ with the variables permuted via $s$.

2. Definition and Elementary Computations

Let $F$ be a commutative domain. Let $q \in F^\times$. At some point, we will assume $q$ is not 1 (Kato Module Section) but as of now it is unimportant.

Definition 2.1. The affine Hecke algebra $\mathcal{H}_{F,n}$ (over $F$) is the unital associative $F$-algebra
generated by the elements $T_1, \ldots, T_{n-1}, X_1^\pm, \ldots, X_n^\pm$ subject to
- The Eigenvalue Relations:
  
  \[(T_i - q)(T_i + 1) = 0\]
- The Laurent Relations:
  
  \[X_iX_j = X_jX_i\]
  \[X_iX_i^{-1} = X_i^{-1}X_i = 1\]
- The Braid Relations:
  
  \[T_iT_j = T_jT_i \text{ if } |i - j| > 1\]
  \[T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}\]
- The Action Relations:
  
  \[T_iX_j = X_jT_i \text{ if } i \neq j, j - 1\]
  \[T_iX_iT_i = qX_{i+1}\]

Note that the last relation also implies that $T_iX_i^{-1}T_i = qX_{i}^{-1}$. 

We also define some useful subalgebras of $\mathcal{H}_{F,n}$.

**Definition 2.2.** The subalgebra generated by the $X_i$ is denoted as $\mathcal{P}_{F,n}$.
The subalgebra of symmetric Laurent polynomials in the $X_i$ is denoted as $\mathcal{Z}_{F,n}$.
The subalgebra generated by the $T_i$ is denoted as $\mathcal{H}^T_{F,n}$. In particular, if $w$ is an element in $S_n$, then since the braid relations hold in $\mathcal{H}_{F,n}$, we can unambiguously define the element $T_w \in \mathcal{H}^T_{F,n}$ by taking any reduced word for $w$.

We end this section with the following useful computational tool. The proof is left as an exercise.

**Lemma 2.3.** Let $f \in \mathcal{P}_{F,n}$. We have a standard action of $S_n$ on $\mathcal{P}_{F,n}$ by permutation of variables. If $s_i$ denotes the transposition $(i, i+1)$, then we have

$$T_i f = (s_i \cdot f) T_i + (q - 1) \frac{f - (s_i \cdot f)}{1 - X_i X^{-1}_{i+1}}$$

Now, if $w$ is a reduced expression of an element of $S_n$, then, using the above relation and induction, we have

$$f T_i = T_i (s_i \cdot f) + (q - 1) \frac{f - (s_i \cdot f)}{1 - X_i X^{-1}_{i+1}}$$

and

$$T_w f = (w \cdot f) T_w + (q - 1) \sum_{w'} f_{w'} T_{w'}$$

and

$$f T_w = T_w (w^{-1} \cdot f) + (q - 1) \sum_{w'} T_{w'} g_{w'}$$

where the summation is over $w'$ is a reduced expression that is contained in $w$ i.e. over $w'$ that are smaller than $w$ in the Bruhat order on reduced words in $S_n$.

3. **Basis Theorem - Bernstein Presentation**

For $\alpha \in \mathbb{Z}^n$, define $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Our goal for this section is to prove the following theorem.

**Theorem 3.1.** The set

$$\mathcal{B} := \{X^\alpha T_w : \alpha \in \mathbb{Z}^n, w \in S_n\}$$

is an $F$-basis for $\mathcal{H}_{F,n}$.

**Proof.** The above lemma shows us that $F \mathcal{B}$ is invariant under left multiplication by the generators $T_i, X_i$ and is hence all of $\mathcal{H}_{F,n}$. Thus, we need to show that $\mathcal{B}$ is linearly independent over $F$. It suffices to show $\mathcal{B}$ is linearly independent in $\mathcal{H}_n$, the affine Hecke algebra with generic $q$ defined over $A = F[q, q^{-1}]$, because non-trivial relations after specializations of $q$ can be lifted to non-trivial relations before specialization. In this case, we show that $\mathcal{B}$ is linearly independent by constructing an $A$-representation of $\mathcal{H}_n$ in which $\mathcal{B}$ maps to a linearly independent set of operators. Secretly, this representation is induced from
the trivial representation of $H_n^T$ but we do not know that until after the basis theorem is proved.

Let $H_n$ act on $A[Y_1^\pm, \ldots, Y_n]$ via

(a) \[ X_i^\pm \cdot f = Y_i^\pm f \]

(b) \[ T_i \cdot f = s_i \cdot f + (q-1) \frac{f - s_i \cdot f}{1 - Y_i Y_{i+1}^{-1}} \]

We first show that actually defines a representation by checking that the defining relations are satisfied. It's immediate that the Laurent relations, the first action relation and the first braid relations hold. Checking that the eigenvalue relations hold is an easy computation that we leave as an exercise. So, we only need to show now that the last action relation holds and that the second braid relations hold. We compute the action relations and leave the braid relations as a similar exercise.

\[ T_i X_i T_i(f) = T_i \left( Y_i(s_i \cdot f) + (q-1)Y_i Y_{i+1} f - (s_i \cdot f) \right) \]
\[ = Y_{i+1} f + (q-1)Y_{i+1} \left( Y_i(s_i \cdot f) - Y_{i+1} f + Y_i (s_i \cdot f) - f \right) \]
\[ + (q-1)^2 Y_i Y_{i+1} \left( f - (s_i \cdot f) + (s_i \cdot f) - (s_i^2 \cdot f) \right) \]
\[ = Y_{i+1} f + (q-1)Y_{i+1} f = qY_{i+1} f = qX_{i+1}(f) \]

as desired.

We now check that the set $B$ maps to $A$-linearly independent operators under this representation. Suppose we have

\[ M = \sum_i c_i B_i = 0 \]

with $c_i \in A$, $B_i = x^{\alpha_i}T_{w_i} \in B$. We claim inductively that $c_i \in (q-1)^j \subseteq A$. The base case of $j = 0$ is obvious. Assume the statement holds for $j-1$. Then, for any $f \in A[Y_1^\pm, \ldots, Y_n^\pm]$, the action of $M$ (modulo $(q-1)^j$) is given by

\[ 0 = \sum_i c_i x^{\alpha_i}(w_i \cdot f) \]

Now, if we let $N$ be bigger than all the $|\alpha_i|$ (which is the sum of the absolute value of its components), then choosing $f = X_1^N \cdots X_n^N$, we see that the monomials

\[ x^{\alpha_i}(w_i \cdot f) = x^{\alpha_j}(w_j \cdot f) \Leftrightarrow i = j. \]

Hence, using both these relations, we see that $c_i \equiv 0 \mod (q-1)^j$, which completes the induction step.

Hence, we have

\[ c_i \in \cap_{j=0}^\infty (q-1)^j = 0. \]

Thus, the set $B$ is linearly independent over $A$, and hence forms a basis, as desired.
We have an obvious corollary of the above theorem.

**Corollary 3.2.** $\mathcal{P}_{F,n}$ is isomorphic to the algebra of Laurent polynomials in $n$ variables over $F$.

$\mathcal{H}_{F,n}$ is isomorphic to the Hecke algebra of type $A_{n-1}$ over $F$.

This completes this section. We now move on to describing the center of the affine Hecke algebra.

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### 4. The Center of the Affine Hecke Algebra

Let $Z_{F,n}$ be the subalgebra of symmetric Laurent polynomials in the $X_i$. Then, we have the following result.

**Theorem 4.1.** $Z(\mathcal{H}_{F,n}) = Z_{F,n}$.

**Proof.** Note that $Z_{F,n}$ clearly commutes with each $X_i$ and it also commutes with each $T_i$ by Lemma 2.3. Hence, $Z_{F,n} \subseteq Z(\mathcal{H}_{F,n})$.

To get the reverse inclusion, we first show that $Z(\mathcal{H}_{F,n}) \subseteq F[X_1^\pm, \ldots, X_n^\pm]$. We use the basis theorem proved before. Suppose

$$f = \sum_i c_i X_0 T_{w_i} \in Z(\mathcal{H}_{F,n}).$$

Let $w_0$ in the decomposition of $f$ be an element of $S_n$ that is maximal in the Bruhat order. Suppose for contradiction that $f \notin \mathcal{P}_{F,n}$. Then, $w_0 \neq e$. Let $j \in \{1, \ldots, n\}$ be such that $w_0(j) \neq j$. Then,

$$X_j f = \sum_i c_i X_j X_0 w_i,$$

which has $w_0$ coefficient $c_0 X_0 X_0$, but by Lemma 2.3, the $w_0$ coefficient of

$$f X_j = \sum_i c_i X_0 w_i X_j$$

is $c_0 X_0^{w_0(j)} X_0^{w_0}$ which is not the same. Hence, by the basis theorem, $X_j f \neq f X_j$, a contradiction. Hence, $Z(\mathcal{H}_{F,n}) \subseteq \mathcal{P}_{F,n}$.

We now show that $Z(\mathcal{H}_{F,n}) \subseteq Z_{F,n}$. By Lemma 2.3, for a Laurent polynomial $f$,

$$T_i f = (s_i \cdot f) T_i + (q - 1) \frac{f - (s_i \cdot f)}{1 - X_i X_{i+1}}.$$

So, if $f \in Z(\mathcal{H}_{F,n})$, then, by the basis theorem,

$$f T_i = (s_i \cdot f) T_i$$

and hence $f$ is symmetric (as $i$ is arbitrary).

□

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5. Kato Modules and Kato’s Theorem

From now on, let $F$ be a field and let $q \in F^\times$ not be 1. Let $\mathcal{H}_{F,n}$ be denoted by $\mathcal{H}_n$ (analogous convention for $\mathcal{P}_n, \mathcal{H}_n^T, \mathbb{Z}_n$). Let $\mathcal{P}_n$-mod be the category of finite-dimensional (over $F$) $\mathcal{P}_n$ modules. We want to understand the $\mathcal{H}_n$-modules of particular central character i.e. modules in which $\mathbb{Z}_n$ acts by each $X_i$ acting as the same scalar. The last section of this talk will give a complete description of such modules. In this section, we merely study the simple objects in this subcategory of $\mathcal{H}_n$-mod. Our goal of this section is thus to prove the following theorem.

**Theorem 5.1.** Let $a \in F^\times$. Take the one-dimension $\mathcal{P}_n$-module $L(a, \ldots, a)$ in which $X_i$ acts as $a$ and define $L(a^n)$ to be the induced $\mathcal{H}_n$-module

$$\text{Ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(L(a, \ldots, a)).$$

Then, $L(a^n)$ is the unique simple module in its central character block.

We first introduce the notion of formal characters on $\mathcal{P}_n$-modules.

**Definition 5.2.** For $\underline{a} = (a_1, \ldots, a_n) \in (F^\times)^n$, we have a one-dimensional representation of $\mathcal{P}_n$ in which $X_i$ acts as $a_i$. These form a complete list of irreducible $\mathcal{P}_n$-modules.

For an arbitrary finite-dimensional $\mathcal{P}_n$-module $M$, let $M_{\underline{a}}$ be the largest submodule whose composition factors are $L(\underline{a})$ or equivalently, the generalized eigenspace in $M$ in which $X_i$ has eigenvalue $a_i$.

We then have the obvious result.

**Lemma 5.3.** For any $M \in \mathcal{P}_n$-mod, we have

$$M = \bigoplus_{\underline{a} \in (F^\times)^n} M_{\underline{a}}.$$

We now define the formal character for a representation $M \in \mathcal{H}_n$-mod.

**Definition 5.4.** For $M \in \mathcal{P}_n$-mod, let $[M]$ be its image in the Grothendieck group. Then, for $M \in \mathcal{H}_n$-mod, we define the formal character of $M$, $\text{ch} M$, to be $[\text{Res}_{\mathcal{P}_n} M]$.

Since $\text{Res}_{\mathcal{P}_n}$ is an exact functor, $\text{ch}$ is a homomorphism from the Grothendieck group of $\mathcal{H}_n$-mod to the Grothendieck group of $\mathcal{P}_n$-mod. The following Lemma will be useful later.

**Lemma 5.5.** Let $\underline{a} = (a_1, \ldots, a_n) \in (F^\times)^n$. Then,

$$\text{ch Ind}_{\mathcal{P}_n}^{\mathcal{H}_n}(L(a, \ldots, a)) = n!L(a, \ldots, a).$$

**Proof.** Fix $\underline{a} = (a_1, \ldots, a_n)$. By the basis theorem, an $F$-basis for $L(\underline{a})$ is given by $\{T_w \otimes 1 : w \in S_n\}$. Put a total order on this basis by refining the Bruhat order on $S_n$. Then, for each $i$ from 1 to $n$, we have by Lemma 2.3

$$X_i(T_w \otimes 1) = (T_w \otimes X_{w^{-1}(i)})(1) + \sum_{w' < w} T_{w'} \otimes g_{w'}(1) = a_{w^{-1}(i)}(T_w \otimes 1) + \sum_{w' < w} T_{w'} \otimes g_{w'}(1).$$
Hence, every $X_i$ acts as an upper triangular matrix in the given basis for $L(a^n)$ with $a$’s on the diagonals. Now, we have an ascending filtration $\{M_l\}$ (as a $P_n$-mod) of $\text{Ind}^H_n L(a)$ where

$$M_l = \bigoplus_{i=1}^{l} F(T_{w_i} \otimes 1).$$

Thus, since the filtration and its associated graded module give the same element in the Grothendieck group, we have by the above computation,

$$\text{ch Ind}^H_n L(a) = \bigoplus_{i=1}^{n!} \text{ch}(M_i/M_{i-1}) = \bigoplus_{i=1}^{n!} L(w_i(a))$$

as desired.

We now define central characters of representations in $H_n$-mod.

**Definition 5.6.** Since $Z(H_n) = Z_n$, for each $\underline{a} = (a_1, \ldots, a_n) \in (\mathbb{F}^\times)^n$ we can define a homomorphism

$$\chi_{\underline{a}} : Z_n \rightarrow F$$

by sending the Laurent polynomial $f(x_1, \ldots, x_n) \mapsto f(a_1, \ldots, a_n)$.

By elementary theory of symmetric functions, we have $\chi_{\underline{a}} = \chi_{\underline{b}}$ if and only if $\underline{b}$ is in the orbit of $\underline{a}$ in the action of $S_n$. We say that $\chi_{\underline{a}}$ is a central character of $H_n$. If $\gamma$ is the orbit in $F^n$ of $\underline{a}$ under the action of $S_n$, we also say that $\gamma$ is a central character of $H_n$.

Now, in any irreducible in $H_n$-mod, $Z_n$ acts via a particular central character. Thus, $H_n$-mod splits up as the direct sum of abelian subcategories corresponding to a particular central character. This gives us the following definition.

**Definition 5.7.** The subcategory of $H_n$-mod in which every object has, as composition factors, irreducibles in which the center $Z_n$ acts via the character $\gamma$ is called the block in $H_n$-mod corresponding to $\gamma$. We denote the block corresponding to $\gamma$ by $H_n$-mod[$\gamma$].

We are now ready to define Kato modules and prove Kato’s theorem regarding these modules. Kato modules are objects that are analogous to Verma modules in Lie theory.

**Definition 5.8.** For $a \in F^\times$, we define the Kato module (with central character $\gamma_a = (a, \ldots, a)$) to be

$$L(a^n) := \text{Ind}^H_n L(a, \ldots, a).$$

By Lemma 5.5, we know that $L(a^n)$ has central character $\gamma_a$ and hence belongs to the block corresponding to this character. Kato’s Theorem then states that $L(a^n)$ is the unique irreducible in its block.

We build up to Kato’s Theorem by first proving the following lemma.

**Lemma 5.9.** Let $a \in F^\times$. Let $L = L(a, \ldots, a)$ (and hence $L(a^n) = H_n \otimes_{P_n} L$). The common $a$-eigenspace of the operators $X_1, \ldots, X_{n-1}$ on $L(a^n)$ is precisely $1 \otimes L$. 

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Proof. By the basis theorem, \( L(a^n) = \bigoplus_{w \in S_n} T_w \otimes L \). We prove by induction that the common \( a \)-eigenspace for \( X_1, \ldots, X_i \) is

\[
\bigoplus_{y \in \langle s_{i+1}, \ldots, s_{n-1} \rangle} T_y \otimes L.
\]

We denote \( \langle s_{i+1}, \ldots, s_{n-1} \rangle \) as \( \Delta_i \). We use the base case of \( i = 0 \), which is vacuously true. So, now suppose that \( i \geq 1 \) and suppose that the common \( a \)-eigenspace for \( X_1, \ldots, X_{i-1} \) is

\[
\bigoplus_{y \in \Delta_{i-1}} T_y \otimes L.
\]

Now, any \( T_w \) for \( w \in \Delta_{i-1} \) can be written as \( T_{w'}T_iT_{i+1} \cdots T_j \) with \( i-1 \leq j \leq n-1 \) and \( w' \in \Delta_i \). Then, by Lemma 2.3 and some computation, we have for any \( v \in L \),

\[
(X_i - a)(T_w \otimes v) = -(q - 1)aT_{w'}T_i \cdots T_{j-1} \otimes v + (*)
\]

where \( (*) \) stands for terms that belong to

\[
\bigoplus_{y' \in \Delta_i, k < j} T_{y'}T_i \cdots T_k \otimes L.
\]

Now, suppose \( z \) is in the common \( a \)-eigenspace of \( X_1, \ldots, X_i \). By the induction hypothesis, we can write, for some fixed nonzero \( v \in L \),

\[
z = \sum_{y \in \Delta_{i-1}} c_y T_y \otimes v = \sum_{w' \in \Delta_i, i-1 \leq j \leq n-1} c_{w', j} T_{w'}T_i \cdots T_j \otimes v.
\]

Choose maximal \( j \) for which \( c_{w', j} \) is nonzero (for some \( w' \)). But then, the above calculation shows that, since \( q \neq 1 \) unless \( j = i-1 \) (i.e. there are only the \( w' \) terms), \( (X_i - a)z \) has a nonzero \( T_{w'} \cdots T_{j-1} \) coefficient. Thus, by the basis theorem, we have \( z \in \bigoplus_{y \in \Delta_i} T_y \otimes L \). This completes the induction step. Since \( \Delta_{n-1} = \{1\} \), we have the desired result.

We finish the section by proving Kato’s Theorem.

**Theorem 5.10.** Let \( a \in F^\times \) and let \( \mu = (\mu_1, \ldots, \mu_r) \) be a composition of \( n \). We define \( H_{\mu} = H_{\mu_1} \otimes \cdots \otimes H_{\mu_r} \) and define \( H_{n-1} \) to be the affine Hecke subalgebra generated by \( X_1^\pm, \ldots, X_{n-1}^\pm \) and \( T_1, \ldots, T_{n-2} \). Then:

1. \( L(a^n) \) is irreducible and it is the only irreducible in its block.
2. All composition factors of \( \operatorname{Res}_{H_{\mu}}(L(a^n)) \) are isomorphic to \( L(a^{\mu_1}) \otimes \cdots \otimes L(a^{\mu_r}) \)

and the socle of \( \operatorname{Res}_{H_{\mu}}(L(a^n)) \) is irreducible.

3. The socle (the sum of all simple submodules) of \( \operatorname{Res}_{H_{n-1}}(L(a^n)) \) is \( L(a^{n-1}) \).

**Proof.** As before, let \( L \) denote \( L(a, \cdots a) \in P_n\text{-mod}. \)
(1) Let $M$ be a nonzero $\mathcal{H}_n$-submodule of $L(a^n)$. Since $L(a^n)$ restricted to $\mathcal{P}_n$ has composition factors all isomorphic to $L$, so does $M$ by Lemma 5.5. Hence, $\text{Res}_{\mathcal{P}_n}(M)$ contains a $\mathcal{P}_n$-submodule $N$ isomorphic to $L$. Now, $\mathcal{P}_n$ acts on $L$ via scalars in which each $X_i$ acts as $a$. Thus, $N$ is contained in $1 \otimes L$, the common $a$-eigenspace of $X_1, \ldots, X_n$. But, we know that $1 \otimes L$ is irreducible as a $\mathcal{P}_n$-module. Hence, $\text{Res}_{\mathcal{P}_n}(M)$ contains a $\mathcal{P}_n$-submodule isomorphic to $L$. Now, $\mathcal{P}_n$ acts on $L$ via scalars in which each $X_i$ acts as $a$. Thus, $N$ is contained in $1 \otimes L$, the common $a$-eigenspace of $X_1, \ldots, X_n$. But, we know that $1 \otimes L$ is irreducible as a $\mathcal{P}_n$-module. Hence, $N = 1 \otimes L$ and hence

$$M \supseteq \mathcal{H}_n(1 \otimes L) = L(a^n).$$

Thus, $L(a^n)$ is irreducible. Now, if $M'$ is any other representation in the same block, by central character considerations, $\text{Res}_{\mathcal{P}_n}(M')$ must contain a $\mathcal{P}_n$-submodule isomorphic to $L$ and hence by Frobenius Reciprocity, $M'$ contains an $\mathcal{H}_n$-submodule isomorphic to $L(a^n)$.

(2) The fact that all composition factors of $\text{Res}_{\mathcal{H}_\mu}(L(a^n))$ are isomorphic to $L(a^{\mu_1}) \boxtimes \cdots \boxtimes L(a^{\mu_r})$ is immediate from unicity of irreducibility of $L(a^{\mu_i})$ and central character considerations via Lemma 5.5. To see that the socle of $\text{Res}_{\mathcal{H}_\mu}(L(a^n))$ is irreducible first note that the $\mathcal{H}_\mu$-submodule $\mathcal{H}_\mu L \cong H_\mu \otimes L$ of $L(a^n)$ is isomorphic to the irreducible

$$L(a^{\mu_1}) \boxtimes L(a^{\mu_r}).$$

Conversely if $M$ is any irreducible $\mathcal{H}_\mu$-submodule of the restriction, then using the same argument as in the proof of (1), we see that $M$ contains $L$ and hence must be $H_\mu L$. Thus, $H_\mu \otimes L$ is the socle, which is hence irreducible.

(3) First note that by part 2, $L(a^n)$ has a unique $\mathcal{H}_{n-1,1}$ submodule $\mathcal{H}_{n-1,1} \otimes L \cong L(a^{n-1}) \boxtimes L(a)$, which is the socle of the restriction of $L(a^n)$. This gives us a one-dimensional contribution of $L(a^{n-1})$ to the socle. However, if $M$ is any other irreducible $\mathcal{H}_{n-1}$-submodule of $L(a^n)$, then $M$ must contain $\mathcal{H}_{n-1} \otimes L$ by the same argument as in (1). Hence, the socle of $\text{Res}_{\mathcal{H}_{n-1}}(L(a^n))$ is isomorphic to $L(a^{n-1})$.

\[\square\]

Part B - Cyclotomic Hecke Algebras of Type A

6. Introduction and Notation

Cyclotomic Hecke Algebras, called Ariki-Koike algebras in the main reference [GJ11], are defined as $q$-deformations of the group algebras of the complex reflection groups of type $G(m, 1, n)$. They can also be viewed as cyclotomic quotients of the affine Hecke algebra. We begin Part B by briefly describing the construction and representation theory of these groups.

Additionally, we define some notation here that will be fixed for the section of the talk on AK-Algebras. This list of notation is for the convenience of the reader. Terms used in the notation section will be defined when introduced later.

Notations and Conventions
R denotes an arbitrary commutative domain with unity of characteristic 0, unless specified otherwise. \( k \) is the field of fractions of \( R \).

(2) For \( q, q_1, \ldots, q_{n-1} \) generic variables, \( A \) is the commutative ring \( R[q^\pm, q_1, \ldots, q_{n-1}] \) and \( K \) is its field of fractions.

(3) We fix an \( m \in \mathbb{Z}_{>0} \). Then, \( W_n \) denotes the complex reflection groups of type \( G(m, 1, n) \).

(4) \( H_{R,n} \) denotes the Ariki-Koike Algebra associated to \( G(m, 1, n) \) over the ring \( R \).

(5) For \( \lambda \) a partition of \( n \), we write \( \lambda \vdash n \).

For \( \lambda \) an \( m \)-partition of \( n \), we write \( \lambda \vdash m n \).

7. Complex Reflection Groups of Type \( G(m, 1, n) \)

**Definition 7.1.** A complex reflection in \( GL(r, \mathbb{C}) \) is a matrix whose 1-eigenspace has dimension \( r - 1 \). In other words, a complex reflection is an automorphism of a complex vector space which fixes some hyperplane pointwise. Note however, that a complex reflection does not have to have order 2.

A complex reflection group is a subgroup of \( GL(r, \mathbb{C}) \) for some \( r \) that is generated by complex reflections.

Finite complex reflection groups have been completely classified by Sheppard and Todd. In their classification, there are 34 exceptional groups and one infinite family of groups \( G(m, p, n) \), where \( m, p, n \) are positive integers. In this talk, we will only care about the complex reflection groups of type \( G(m, 1, n) \). We now give 3 realizations of this group:

1. \( G(m, 1, n) \) is the wreath product \( \mathbb{Z}/m\mathbb{Z} \wr S_n \), which is the group \((\mathbb{Z}/m\mathbb{Z})^n \rtimes S_n\) with the symmetric group acting via permutation of coordinates.

2. We can also define \( G(m, 1, n) \) using generators and relations. There is a presentation of \( G(m, 1, n) \) with generators \( S = \{ s_i : i = 0, \ldots, n - 1 \} \) and relations

   - \( s_0^m = 1 \)
   - \( s_i^2 = 1 \) for \( i > 0 \).
   - \( s_i s_j = s_j s_i \) if \( |i - j| > 1 \).
   - \( s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \).
   - \( s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \) if \( i \geq 1 \).

   The last 3 relations are called the braid relations in \( G(m, 1, n) \).

3. Finally, we can realize \( G(m, 1, n) \) inside \( GL(n, \mathbb{C}) \) as the subgroup of monomial matrices with entries that are \( m \)th roots of unity. Here, for some primitive \( m \)th root of unity \( \zeta_m \), we have \( s_0 = \zeta_m E_{1,1} + \sum_{j \neq i} E_{jj} \) and \( s_i = E_{i,i+1} + E_{i+1,i} + \sum_{j \neq i, i+1} E_{jj} \).

   It is this last definition that makes it clear that \( G(m, 1, n) \) is a finite complex reflection group.

From now on, fix a positive integer \( m \) and let \( W_n \) denote \( G(m, 1, n) \). As a useful fact, we note that \( |W_n| = m^n n! \).
We now define some combinatorial objects that generalize the notion of a partition and that will be useful in the representation theory of \(W_n\) and the associated \(AK\)-algebra.

**Definition 7.2.** We call \(\lambda = (\lambda^1, \ldots, \lambda^m)\) an \(m\)-partition of \(n\), if each \(\lambda^i\) is a partition of \(\{\lambda^i - 1 + 1, \ldots, \lambda^i - 1 + \lambda^i\}\) and \(\sum_{i=1}^m |\lambda^i| = n\).

Note that in \(m\)-partitions, we allow some of the \(\lambda^i\) to have size 0.

Let \(R\) now be a commutative domain of characteristic 0 and let \(k\) be its fraction field. Assume that \(R\) contains the \(m\)th roots of unity. Then, it turns out that \(k\) is a splitting field for \(W_n\) and that the irreducible representations of \(W_n\) over \(k\) are indexed by \(m\)-partitions of \(n\). We give a brief description of these representations and leave the verification of the details as an exercise that can be looked up in [Mac95, 1, Appendix B] if needed.

First, we define \(m\) 1-dimensional representations of \(W_n\) via the \(m\) irreducible characters of \(\mathbb{Z}/m\mathbb{Z}\). Let \(\zeta_m\) be a fixed primitive \(m\)th root of unity. We then define the representation \(\sigma_k\) by sending \(s_0 \mapsto \zeta_m^k \in k\) and \(s_i \mapsto 1, i > 0\).

Let \(\lambda = (\lambda^1, \ldots, \lambda^m)\) now be an \(m\)-partition of \(n\) and suppose \(n_i = |\lambda^i|\). For each \(i\), we can use the natural projection \(W_{n_i} \to S_{n_i}\) to extend the irreducible Specht module \(E_{\lambda^i}\) of \(S_{n_i}\) to an irreducible representation of \(W_{n_i}\). Then, since we have \(W_n = W_{n_1} \times \cdots \times W_{n_m} \subseteq W_n\), we can define the representation \(E_{\lambda} := \text{Ind}_{W_n}^{W_{n_1}}((E_{\lambda^1} \otimes \sigma_1) \boxtimes (E_{\lambda^2} \otimes \sigma_2) \boxtimes \cdots \boxtimes (E_{\lambda^m} \otimes \sigma_m))\).

We now have the following theorem:

**Theorem 7.3.** The above procedure gives a complete list of non-isomorphic simple \(W_n\) representations, that is,

\[
\text{Irr}_k(W_n) = \{E_{\lambda} : \lambda \vdash m n\}.
\]

We next describe the branching rule for restriction of representations from \(W_n\) to \(W_{n-1}\). To do so, we generalize the notion of Young Tableaux to \(m\)-partitions.

**Definition 7.4.** For \(\lambda = (\lambda^1, \ldots, \lambda^m) \vdash m n\), we define the Young tableau \([\lambda]\) of \(\lambda\) to be an \(m\)-tuple of Young tableaux \(([\lambda^1], \ldots, [\lambda^m])\).

We define the set of addable (resp. removable) boxes, \(\text{add}(\lambda)\) (resp. \(\text{rem}(\lambda)\)), to be the union of the set of addable (resp. removable) boxes in each component tableau.

For \(x \in \text{add}(\lambda)\) (resp. \(\text{rem}(\lambda)\)), we define \([\lambda + \{x\}]\) (resp. \([\lambda - \{x\}]\)) as the \(m\)-partition obtained by adding (resp. removing) the box \(x\).

Then, we have the following branching rule in \(W_n\):

**Theorem 7.5.** For all \(\lambda \vdash m n\), we have

\[
\text{Res}_{W_{n-1}}^{W_n}(E_{\lambda}) = \bigoplus_{\mu} E_{\mu}
\]

where the sum is taken over all \([\mu]\) that are obtained by removing a box from \([\lambda]\).
This finishes our discussion of the representation theory of the complex reflection groups $W_n$. We now move on to the associated Cyclotomic Hecke Algebra.

8. Cyclotomic Hecke Algebra: Definition and Examples

From now on, we fix the commutative domain $R$ containing $\mathbb{C}$ and let $k$ be its field of fractions. We define the cyclotomic Hecke algebra $H_{R,n}$ as a deformation of the group algebra $R[W_n]$. Let $q, q_1, \ldots, q_m \in R^\times$. Then:

**Definition 8.1.** The CH-algebra $H_{R,n} = H_{R,n}(q, q_1, \ldots, q_m)$ is defined as the unital associative $R$-algebra generated by the elements $T_0, \ldots, T_{n-1}$ subject to

- The Eigenvalue relations:
  
  \[(T_0 - q_1) \cdots (T_0 - q_m) = 0\]
  
  \[(T_i - q)(T_i + 1) = 0 \text{ for } i > 0.\]

- The Braid Relations:
  
  \[T_i T_j = T_j T_i \text{ if } |i - j| > 1\]
  
  \[T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } i > 0\]
  
  \[T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0.\]

We give a few examples of $H_{R,n}$ for particular values of $q, q_i$.

**Example 8.2.** We give 3 examples here and leave the verification of the details as an exercise.

1. Suppose $R$ contains the primitive $m$th root of unity $\zeta_m$. If $q = 1$ and $q_j = \zeta_j^i$. Then, $H_{R,n} \cong R[W_n]$. In particular, if instead of $R$, we use its field of fractions $k$, then the corresponding Ariki-Koike algebra is split semisimple.

2. Suppose $l = 1$ and suppose $q$ has a square root in $R$. Then, $H_{R,n}$ is the Hecke algebra over $R$ of type $A_{n-1}$.

3. Suppose $l = 2$ and suppose $q, q_1, q_2$ have square roots in $R$. Then, $H_{R,n}$ is the Hecke algebra over $R$ of type $B_n$.

So we see that many interesting algebras are simply special cases of $H_{R,n}$, which gives us enough reason to want to understand its structure and representation theory.

We end the section with the following important remark:

**Remark.** There is a very useful realization of the cyclotomic Hecke algebra as a quotient of the affine Hecke algebra $\mathcal{H}_{R,n}$ by the two sided ideal generated by $(X_1 - q_1) \cdots (X_1 - q_m)$. From the definition of the defining relations for each algebra, it is not difficult to see that there is a surjective algebra homomorphism

\[\Phi : \mathcal{H}_{R,n} \to H_{R,n}\]
given by sending $T_i$ to $T_i$ for $i > 0$ and sending $X_1$ to $T_0$. This gives additional motivation to the study of cyclotomic Hecke algebras because any finite dimensional representation of the affine Hecke algebra factors through a cyclotomic quotient.

9. Jucys-Murphy Basis of Cyclotomic Hecke Algebras

We now define some special elements of $H_{R,n}$ that are called Jucys-Murphy elements. For $j = 1, \ldots, n$, define

$$L_j = q^{1-j}T_{j-1} \cdots T_0 T_1 \cdots T_{j-1}.$$  

**Remark.** If we assume $m = 1$ and specialize at $q = 1$, then $H_{k,n} \cong k[S_n]$. However, under this specialization, the Jucys-Murphy elements defined here are not the same as the Jucys-Murphy elements defined classically.

**Remark.** The Jucys-Murphy elements defined above can also be defined as $L_j = \Phi(X_j)$, where $\Phi$ is the map defined in the terminal remark of the previous section. All of the following identities for the Jucys-Murphy elements can thus be proved by proving them at the level of the affine Hecke algebra.

We note down some useful properties of the Jucys-Murphy elements and leave the proofs as an exercise (in applying the braid relations or using the affine Hecke algebra relations).

**Proposition 9.1.** For $L_i$ defined as above, we have:

1. $L_i$ commutes with $L_j$.
2. $T_i$ commutes with $L_j$ if $j \neq i, i+1$.
3. $T_i$ commutes with $L_iL_{i+1}$ and $L_i + L_{i+1}$.
4. For all $a \in R$ and $i \neq j$, $T_i$ commutes with $\prod_{1 \leq l \leq j}(L_l - a)$.

Now, as the generators $s_i$ for $i > 1$ satisfy the relations of the Hecke algebra of type $A_{n-1}$, we can uniquely define $T_w$ for any $w \in S_n$. Then, as a corollary of the proposition above, we have the following Lemma.

**Lemma 9.2.** The following identities hold in $H_{R,n}$:

1. For $i \geq 1$,

$$T_i L_{i+1}^k = q^k L_{i+1}^k T_i + (q-1) \sum_{j=1}^{k} q^{1-j} L_{i+1}^{j-1} L_i^{k-j+1}.$$

2. For $i \geq 1$,

$$T_i L_i = q^{-k} L_{i+1}^k T_i + (q^{-1} - 1) \sum_{j=1}^{k} q^{1-j} L_i^{k-j} L_{i+1}^j.$$

We leave the proof of the Lemma as an exercise. We note that the exact formulas are not very important. The key idea is that $T_i$ applied to either $L_{i+1}^k$ or $L_i^k$ interpolates between $L_{i+1}$ and $L_i$ while keeping the total power constant.

We now define a distinguished set $X \subseteq H_{R,n}$ of size $m^n n!$ as follows.

**Definition 9.3.**

$$X := \{L_1^{c_1} \cdots L_n^{c_n} T_w : w \in S_n, 0 \leq c_i \leq m - 1\}.$$
Our goal in this section is to prove that $X$ is a basis for $H_{R,n}$ over $R$. With Lemma 9.2 in hand, we can prove the first part of this goal as the following theorem.

**Theorem 9.4.** The set $X$ spans $H_{R,n}$ over $R$.

*Proof.* Since $R\langle X \rangle$ contains the unit element, it suffices to show that this set is stable under left multiplication by $H_{R,n}$. For this, it suffices to show that $R\langle X \rangle$ is stable under left multiplication by each $T_i$. Fix some $0 \leq c_1, \ldots, c_n \leq m - 1$ and some $w \in S_n$. Let $c$ be the $m$-tuple $(c_1, \ldots, c_m)$ and let $L_{c,w}$ denote the obvious Jucys-Murphy element. Then, we have

$$T_0 L_{c,w} = L_{c+1,w}.$$  

For $c < m - 1$, this gives another element of $X$. For $c = m - 1$, we can use the Eigenvalue relation to write this as a sum of $L_{a,w}$ with $0 \leq a \leq m - 1$. Hence, $R\langle X \rangle$ is stable under left multiplication by $T_0$. Now, fix some $0 < i < n$. Then, Proposition 9.1 implies that

$$T_i L_{c,w} = L_i^{c_1} \cdots T_i L_i^{c_i} L_{i+1}^{c_{i+1}} \cdots L_n^{c_n} T_i L_{c,w}.$$  

Hence, again using the same proposition, it suffices to show that for arbitrary $0 \leq u, v < l$, we can write $T_i L_i^u L_{i+1}^v$ as a sum of $L_i^u L_{i+1}^v T_i$ and $L_i^v L_{i+1}^u$ with $a, b, a', b' < l$. We prove this fact in the case with $u > v$. The proof in the other case follows very similarly. So, assume $u > v$. Then, using Proposition 9.1 and Lemma 9.2, we have for $X = (L_i L_{i+1})^v$,

$$T_i L_i^u L_{i+1}^v = T_i L_i^{u-v} X + \sum_{j=1}^{u-v} L_i^{u-v-j} L_{i+1}^j X,$$

with the scalars suppressed in the above equation. Since $L_i, L_{i+1}$ commute, we have the desired expression. 

$\square$

We can now state some corollaries of the above theorem.

**Corollary 9.5.** Over any field $\mathbb{K}$, $H_{\mathbb{K},n}$ has dimension at most $n^m n!$

**Corollary 9.6.** To prove that $X$ is linearly independent over $R$, it suffices to show linear independence in the generic case i.e. to prove that $X$ is linearly independent in $H_{A,n}$ where $A = R[q_1^\pm, \ldots, q_m^\pm]$ is the polynomial ring in $q_1^\pm, q_m^\pm$ over $R$ and the parameters for the cyclotomic Hecke algebra are chosen to be the generic variables.

*Proof.* Let $q^\pm, q_1, \ldots, q_m$ now be indeterminate variables and let $A = R[q^\pm, q_1, \ldots, q_m]$. Suppose, for some $X_1, \ldots, X_l \in X$ we have

$$\sum_i c_i X_i = 0$$

with $c_i \in R$ not all 0, where we view $R$ as an arbitrary specialization of $A$. Let $b_i$ be a lift of $c_i$ in $A$. Then, if $q, q_j$ specialize respectively to $\epsilon, \epsilon_j$, then we have

$$\sum_i b_i X_i \in A(q - \epsilon, q_j - \epsilon_j)A.$$  

But since the elements of $X$ span $H_{A,n}$ over $A$ (since nothing special about $R$ was used in the previous theorem), we can rewrite the above relation as the equation

$$\sum_i b_i X_i = \sum_{i'} d_{i'} X_{i'}$$
which gives us a nontrivial relation over $A$ (it’s non-trivial because the $d_i'$ must map to 0 under specialization so they can’t all be the same as the $b_i$.) Thus, it suffices to prove linear independence in the generic case i.e. over $A$.

Let $A$ now be defined in the above corollary and let $K$ be its field of fractions. Proving linear independence of $X$ over $A$ is the same as proving linear independence of $X$ over $K$. Thus, we let $H_n$ now denote $H_{K,n}$ and we prove the statement of linear independence by explicitly constructing a large enough set of irreducible representations of $H_n$ over $K$ and then using a dimension counting argument.

9.1. Irreducible Representations of $H_n$. The full details of the construction of the irreducible representations of $H_n$ is purely technical and is hence left to the appendix of the notes. Here, we merely highlight the salient details of the construction.

Definition 9.7. Let $\lambda \vdash_m n$. We define a standard Young tableau of shape $\lambda$ to be an enumeration from 1 to $n$ of the boxes of the Young diagram of $\lambda$ such that each component tableau is enumerated in a standard manner i.e increasing in rows and columns. For a fixed $\lambda$, define $V_\lambda$ be the formal $K$-linear span of all standard Young tableaux of shape $\lambda$.

It is possible to define an action of $H_n$ on $V_\lambda$. The full details of this action are left to the appendix. The main properties that we will use are:

1. For a standard Young tableau $P$ of shape $\lambda$, $T_0$ acts as the scalar $u_i$ where $i$ the index of the component of $P$ in which 1 appears.
2. Fix a standard Young tableau $P$ of shape $\lambda$. For $i > 1$, if swapping $i$ and $i + 1$ in $P$ does not give a valid standard Young tableau, then $T_i$ acts as a nonzero scalar. If swapping $i$ and $i + 1$ in $P$ gives a standard Young tableau $Q$, then $T_i(P)$ is a linear combination of $P$ and $Q$, with nonzero $Q$ coefficient.

We now come to the main result of this section.

Theorem 9.8. The above action of $H_n$ on $V_\lambda$ gives $V_\lambda$ the structure of an absolutely irreducible representation of $H_n$. Additionally, if $\lambda \neq \mu$, then $V_\lambda \not\cong V_\mu$ as representations of $H_n$.

Proof. Verifying that each $V_\lambda$ is actually a representation of $H_n$ is a tedious exercise in checking that the eigenvalue and braid relations hold. This can be looked up in [AK94, Thm 3.7]. We assume this fact here and just prove the absolute irreducibility and inequivalence of $V_\lambda$ for distinct $\lambda$. For this, we induct on $n$.

The base case of $n = 1$ is obvious. So, we assume the statement holds for $n - 1$. We first prove absolute irreducibility of $V_\lambda$ i.e. that $V_\lambda \otimes K$ is irreducible. Note that $T_0, \ldots, T_{n-2}$ generate a subalgebra $G_{n-1}$ of $H_n$ that is isomorphic to a quotient subalgebra of $H_{n-1}$. Hence, via restriction and pullback, we can view $V_\lambda \otimes K$ as a representation of $H_{n-1}$.

Now, suppose there are $l$ removable boxes in the diagram corresponding to $\lambda$ and let $V_\lambda^{(i)}$ be the subspace of $V_\lambda$ whose basis is given by the standard Young tableaux in which $n$ is in removable box $i$. Then, as a representation of $H_{n-1}$, $V_\lambda$ breaks up as

$$V_\lambda \otimes K = \bigoplus_{i=1}^{l} V_\lambda^{(i)} \otimes K$$
because $T_0, \ldots, T_{n-2}$ do not affect the position of $n$. But, if $\mu$ is the $m$-partition of $n - 1$ which is obtained by removing removable box $i$ from $\lambda$, then, since the action of $T_i$ for $i < n - 1$ does not depend on the position of $n$, the as representations of $H_{n-1}$, we have

$$V^{(i)}_{\lambda} \cong V_{\mu}.$$ 

Thus, by induction, each $V^{(i)}_{\lambda} \otimes K$ is irreducible and for distinct $i, j$, the corresponding irreducibles are nonisomorphic. So, now, let $W$ be a nonzero $H_n$ subrepresentation of $V_{\lambda} \otimes K$. By restricting to $G_{n-1}$, we see that $W$ must contain some $V^{(i)}_{\lambda} \otimes K$. We need to show that it contains all $V^{(j)}_{\lambda} \otimes K$, and to do so, by distinct irreducibility over $G_{n-1}$ of the latter, we note that it suffices to show that $W$ intersects each $V^{(j)}_{\lambda} \otimes K$ nontrivially.

Pick some standard Young tableau $T_\rho$ of shape $\lambda$ such that $n$ is in removable box $i$ and such that $n - 1$ is in removable box $j \neq i$. Let $T_\rho'$ be the SYT with $n, n - 1$ swapped. Then,

$$T_{n-1}(T_\rho \otimes 1) = \lambda_1 T_\rho + \lambda_2 T_\rho'$$

with $\lambda_2 \neq 0$. This implies that $T_\rho' \otimes 1 \in W$. Hence, for each $j$, $W$ intersects $V^{(j)}_{\lambda} \otimes K$ nontrivially and hence contains it. Thus, $W = V_{\lambda} \otimes K$, which is hence irreducible.

All that remains is to show that $V_{\lambda} \not\cong V_{\eta}$ is $\lambda \neq \eta$. But this follows, after restricting to $G_{n-1}$, from the fact that the diagrams obtained by removing one box from $\lambda$ is not the same set as the diagrams obtained by removing one box from $\eta$ unless $\lambda = \eta$. 

This theorem now completes the basis theorem that we desired.

**Theorem 9.9.** The subset $X$ of $H_n$ defined above as

$$X = \{ L_{c_1}^1 \cdots L_{c_w}^w : w \in S_n, 0 \leq c_i \leq m - 1 \}$$

is Linearly Independent over $K$. Additionally, $H_n$ is semisimple.

**Proof.** Since we know that $X$ spans $H_n$ over $K$, for the set to be linearly independent, it suffices to show that the $K$-dimension of $X$ is at least $m^n n!$. If $f_\lambda$ is the dimension of the irreducible $V_\lambda$, then we know that

$$\dim H_n \geq \sum_\lambda f_\lambda^2.$$ 

But, the term on the right is purely combinatorial, and hence is equal to the dimension of $\mathbb{C}[W_n]$, since the irreducibles for that algebra can be constructed in exactly the same way by Theorem 7.3. Hence,

$$\dim H_n \geq m^n n!$$

as desired. This shows that $\dim H_n = m^n n!$, which implies that $H_n$ is semisimple, as its radical must then be trivial.

The proof of Theorem 9.8 and the basis theorem above also give us the branching rule for $H_n$. We first have the following obvious corollary.
Corollary 9.10. The subalgebra of $H_n$ generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the Hecke algebra of type $A_{n-1}$ and the subalgebra generated by $T_0, \ldots, T_{n-2}$ is isomorphic to $H_{n-2}$. In fact, this works over any ring, and not just in the generic case.

Then, the branching rule from representations of $H_n$ to representations of $H_{n-1}$ is given by

**Corollary 9.11.** As an $H_{n-1}$ representation, we have

$$V_\lambda \cong \bigoplus_{\mu} V_\mu$$

where the sum is taken over $m$-partitions $\mu$ of $n-1$ that are obtained from $\lambda$ by removing one box.

We end the section with the following remark that describes a sufficient condition for specializations of $H_n$ to be semisimple. If we look at the construction of the irreducibles given above, we see that as long as we specialize at values of $q, q_i$ such that for every $d$ from $-n$ to $n$ and for every $i, j$ from 1 to $m$, we have

$$q^d q_i q_j \neq 1$$

then the matrix given in the definition of the irreducible $V_\lambda$ will be well-defined for any $\lambda$. In this case, the specialization of $H_n$ will be semisimple, and the irreducibles will be given by specializations of the $V_\lambda$, with the same branching rule.

10. **Symmetric Structure on $H_n$**

In this section, we return to working over arbitrary $\mathbb{C}$-algebras $R$ and arbitrary unit values of $q_i, q$. From this section on, $H_n$ denotes the cyclotomic Hecke algebra over $R$.

We let $X$ be the Jucys-Murphy basis defined as before. For $c \in \mathbb{Z}^n$ with $0 \leq c_i \leq m-1$, define $L^c = L_1^{c_1} \cdots L_n^{c_n}$. Then, we have an $R$-linear map $\tau : H_n \to R$ determined by

$$\tau(L^c T_w) = \begin{cases} 1 & \text{if } c = 0, w = 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

We want to show that the $R$ bilinear form $\nu$ on $H_n$ defined by $\nu(x, y) = \tau(xy)$ is symmetric, nondegenerate and satisfies $\nu(x, zy) = \nu(xz, y)$ for all $x, y, z \in H_n$. The third property is obvious so we focus only on symmetry and non-degeneracy. To prove these properties, we will first construct an alternative $R$ basis for $H_n$ based on reduced words in $W_n$ and then reformulate $\tau$ in that basis.

10.1. **Reduced Words in $W_n$.** We begin with the following definitions.

**Definition 10.1.** A word in $W_n$ is just a word in the alphabet $S = \{s_0, \ldots, s_{n-1}\}$.

We define the length of a word $s_i_1 \ldots s_i_n$ to be $n$ and we say that a word representing $w \in W_n$ is reduced if it has minimal length.

We say that two words are braid equivalent if one can be transformed into the other by using
only the braid relations in \( W_n \).
We sat that two words are weakly braid equivalent if one can be transformed into the other by using the braid relations and one extra relation

\[
s_1 s_0^a s_1 s_0^b = s_0^bs_1 s_0^a s_1
\]
for each \( a, b \in \mathbb{Z} \).

It can be checked that the last relation holds in \( W_n \) but is not a consequence of just the braid relations. We next define specific elements in \( W_n \) that will be useful in studying braid equivalent and weak braid equivalence of reduced words.

**Definition 10.2.** For non-negative integers \( k, a \), we define

\[
l_{k, a} = \begin{cases} \ s_{k-1} \cdots s_1 s_0^a & a > 0 \\ 1 & a = 0 \end{cases}
\]

The following Lemma is now an easy exercise in induction.

**Lemma 10.3.**

(a) For any \( i \) from 1 to \( n \) and any \( a \geq 0 \), we have up to braid equivalence

\[
s_i l_{k, a} = \begin{cases} \ l_{k, a} s_i & i > k \ or \ a = 0 \\ l_{k+1, a} & i = k, a \neq 0 \\ l_{k-1, a} & i = k - 1, a \neq 0 \\ l_{k, a} s_i & i < k - 1, a \neq 0 \end{cases}
\]

(b) For \( a, b > 0 \), we have up to weak braid equivalence

\[
l_{k, a} l_{k, b} = \begin{cases} \ l_{k-1, b} l_{k, a} s_1 & k > 1 \\ l_{1, a+b} & k = 1 \end{cases}
\]

(c) For \( a, b, k > 1 \) and \( m \geq 0 \), we have up to weak braid equivalence

\[
l_{k+m, a} l_{k, b} = l_{k-1, b} l_{k+m, a} s_1.
\]

It turns out that unlike in \( S_n \), two reduced words for the same element of \( W_n \) are not braid equivalent. However, they are weakly braid equivalent. We prove this in the following theorem, carefully marking where we use the weak braid relation because the weak relations get deformed on passing to the cyclotomic Hecke algebra.

**Theorem 10.4.** Let

\[
x = x_1 \cdots x_k, \ x_i \in S
\]
be a reduced word for \( x \in W_n \). Then, \( x \) is braid equivalent to an expression of the form

\[
x = l_{k_1, a_1} \cdots l_{k_r, a_r} w, \ w \in S_n, a_i > 0.
\]

and \( x \) is weakly braid equivalent to an expression of the form

\[
l_{1, a'_1} \cdots l_{n, a'_n} w', \ w' \in S_n, a'_i \geq 0
\]

with \( \sum_j a'_j = \sum_i a_i \).
Proof. Note that it suffices to prove the first claim, as the consequence is immediate from (c) in Lemma 10.3. We prove the statement by inducting on the length of $x$, with the base case being trivial. Suppose the reduced expression for $x$ is of the form

$$x = w_1 s_0^{a_1} \cdots w_l s_0^{a_l} w_{l+1}$$

for $w_i$ a reduced word in $S_n$. Then, by using the braid relations in $S_n$, we can assume

$$w_1 = s_{k_1-1} \cdot s_1 w'_1$$

with $w'_1 \in \langle s_2, \ldots, s_{n-1} \rangle$. Since $w'_1$ commutes with $s_0$, we have under braid equivalence

$$x \equiv l_{k_1, a_1} x'.$$

Since $l(x') < l(x)$ (as $a_1 > 0$), we are done by induction. 

Corollary 10.5. The second set of expressions in the above theorem give us a complete set of reduced expressions for $W_n$, with each representative corresponding to a different element.

We now connect reduced words in $W_n$ with reduced words in $H_{A,n}$.

Definition 10.6. For a reduced expression $x = x_1 \cdots x_k$ of $x \in W_n$, define $T_k = T_{x_1} \cdots T_{x_k}$.

Additionally, in analogy with the elements $l_{k,a}$ in $W_n$, define the elements

$$L_{k,a} = \begin{cases} T_{k-1} \cdots T_i T_0^a & a > 0 \\ 1 & a = 0 \end{cases}.$$

In $H_{A,n}$, the braid relations still hold but the weak braid relations, and hence the relations in Lemma 10.3 are $q$-deformed as follows. Again, the proof is left as an exercise and is very similar to the proof in the undeformed case.

Lemma 10.7. (a) For any $a, b \geq 0$, we have

$$T_1 T_0^a T_1 T_0^b = T_0^a T_1 T_0^b T_1 + q(q - 1) \sum_{i=1}^{b} T_0^{a+b-i} T_1 T_0^i - T_0^i T_1 T_0^{a+b-i}.$$

(b) For any $i$ from 1 to $n - 1$ and $1 \leq k \leq n$,

$$T_i L_{k,a} = \begin{cases} L_{k,a} T_i & i > k, a = 0 \\ L_{k+1,a} & i = k, a \neq 0 \\ q L_{k-1,a} + (q - 1) L_{k,a} & i = k - 1, a \neq 0 \\ L_{k,a} T_{i+1} & i < k - 1, a \neq 0. \end{cases}$$

(c) For $a, b, m > 0, k > 1$, we have

$$L_{k+m,a} L_{k,b} = L_{k-m+1,b} L_{k,a} T_1 + (q - 1) \sum_{i=1}^{b} L_{k-1,a+b-i} L_{k+m,i} - L_{k-1,i} L_{k+m,a+b-i}.$$

The exact expression is unimportant. What is useful is that the sums of the second index remains the same for each monomial term.
Here’s the point of all these annoying calculations. Our goal here is to define a linear form on $H_n$ as the coefficient of $T_e$ when a complete set of reduced expressions is used as a basis for $H_n$. There are, however, two obstructions for this form to be well defined.

1. If $x, x'$ are two reduced expressions for the same word, then we need $T_x - T_{x'}$ to have 0 $T_e$ coefficient.
2. For any complete representative set $\Delta$ of reduced expressions of elements of $W_n$, we need the set

$$X_\Delta := \{T_x : x \in \Delta\}$$

to give us an $R$-basis for $H_n$.

We prove both these facts as corollaries of the following theorem.

**Theorem 10.8.** Let $x, x'$ be reduced expressions for the same element in $W_n$. Then,

$$T_x - T_{x'} \in \sum_{y \notin S_n, \ 0 < l(y) < l(x)} A T_y$$

i.e. their difference involves expressions of smaller length that contain nonzero number of $T_0$ terms.

**Proof.** If $x$ is a reduced expression involving only the $S_n$ generators, then $x'$ must also only involve the $S_n$ generators and hence this follows from Matsumoto’s lemma. So, we assume $x \notin S_n$. Now, since the braid relations still hold in $H_n$, by the proof of Theorem 10.4, we can write

$$T_x = L_{k_1, a_1} \cdots L_{k_r, a_r} T_w, \ w \in S_n, a_i > 0$$

with $r > 0$ i.e. that there is a $T_0$ term. Additionally, without loss of generality, we can assume that

$$T_{x'} = L_{1, a'_1} \cdots L_{n, a'_n} T_{w'}$$

with $\sum_i a'_i = \sum_i a_i$. Now, to move $T_x$ to the normalized $T_{x'}$, we repeatedly use the relations in Lemma 10.7 (c). But for any terms that appear in that Lemma, apart from $T_{x'}$, we have

1. Smaller length.
2. Same total second $L$-index.

The second condition implies that since $\sum a_i > 0$ to begin with, any term that appears in

$$T_x - T_{x'}$$

must involve a $T_0$ term somewhere. Thus, none of these terms lie in $T_{S_n}$. Hence, we have the desired result

$$T_x - T_{x'} \in \sum_{y \notin S_n, \ 0 < l(y) < l(x)} A T_y.$$

□

**Corollary 10.9.** Let $\Delta$ be a complete set of representatives of reduced expressions for elements of $W_n$. Then, $X_\Delta := \{T_x : x \in \Delta\}$ is an $R$-basis for $H_n$. 

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Proof. By dimension considerations, it suffices to prove that $X_\Delta$ spans $H_{A,n}$ over $A$. We prove by induction that $AX_\Delta$ contains $T_y$ for all words (not necessarily reduced) $y$ of length $n$. For $n = 0$, this is obvious, as $X_\Delta$ must contain $T_e$. Suppose it holds for all words of length $n-1$ or less and let $y$ be a word of length $n$.

Now, if $y$ is not reduced, then using the braid and eigenvalue relations we can write $T_y$ as a sum of $T_y'$s with $y'$ all of smaller length. Thus, we can assume that $y$ is reduced. For reduced $y$, if $y \in S_n$, then $T_y$ is in $X_\Delta$, as all reduced word representatives give the same element in $H_n$ (Matsumoto’s Lemma). If $T_y$ is not in $S_n$, pick some $T_x \in X_\Delta$ with $x$ reduced corresponding to $y$ and then use the previous theorem.

\[ \square \]

Corollary 10.10. Let $\Delta$ now be an arbitrary system of representatives and let $X_\Delta$ be as before. Note that $T_e \in X_\Delta$ necessarily. Then, the linear form $\tau': H_{A,n} \to A$ determined by

\[ \tau'(T_x) = \begin{cases} 1 & x = e \\ 0 & \text{otherwise} \end{cases} \]

is independent of the choice of $\Delta$.

Our goal now is to show that $\tau'$ is symmetric, non-degenerate and that $\tau' = \tau$. We prove non-degeneracy only in the Hecke algebra defined over $A = \mathbb{R}[q^\pm, q_i^\pm]$. Non-degeneracy holds for arbitrary specializations of the cyclotomic Hecke algebra but the proof is purely technical and can be looked up in [MM98].

Theorem 10.11. The bilinear form $\sigma'((x, y)) = \tau'(xy)$ from $H_{A,n} \times H_{A,n} \to A$ is non-degenerate.

Proof. It suffices to show that $\sigma'$ is non-degenerate after specializing at particular values of $q, q_i$ (by discriminant considerations). Pick $q = 1$ and $q_i = \zeta^i$, where $\zeta$ is a primitive $m$th root of unity. Under this specialization, $H_{R,n}$ becomes the group algebra $R[W_n]$ (we assume $R$ is a field by taking its field of fractions if necessary) and the form $\tau'$ becomes the standard form on the group algebra that picks out the coefficient of the identity. This form is obviously non-degenerate. Hence, $\tau'$ is non-degenerate over $A$. \[ \square \]

Symmetry requires a little more work.

Theorem 10.12. $\sigma'$ is a symmetric form.

Proof. Since $\sigma'$ is independent of the choice of $\Delta$, we choose $\Delta$ consisting of reduced words in the normalized form described in the second part of Theorem 10.4. We need to show that for every $x \in \Delta$ and every word $y$,

\[ \tau'(T_xT_y) = \tau'(T_yT_x). \]

By induction on the length of $y$, we can assume $T_y = T_i$. By induction on the length of $x$, we can assume it holds for all $T_{x'}$ of smaller length. We now have two cases.

Case $i \geq 1$: Write $T_x$ as

\[ T_x = L_{1,a_1} \cdots L_{n,a_n} T_w \]

for $w \in S_n$. Then, it is clear that

\[ T_x = L_{1,a_1} \cdots L_{n,a_n} T_w \]

for $w \in S_n$. Then, it is clear that

\[ \tau'(T_xT_y) = \tau'(T_yT_x). \]
In the case where it is nonzero, \( T_x = T_i \) and hence

\[
\tau'(T_xT_i) = \tau'(T_iT_x) = \tau'(T_i^2).
\]

So, suppose now that \( T_x \neq T_i \). Then, by a similar proof to the proof of Theorem 10.4, the word \( x \) is also weakly braid equivalent to a reduced word of the form

\[
x' = w'l_{n,a'_n} \cdots l_{1,a'_1}.
\]

Hence, by Theorem 10.8

\[
T_x = T_{x'} + T
\]

where \( T \) is in the span of elements in \( X_\Delta \) of smaller length that do not correspond to words in \( S_n \) and \( x' \neq s_i \) as \( x' \not\in S_n \). Thus, since \( \tau'(TT_i) = 0 \) by the above argument, and \( \tau'(T_iT_{x'}) = 0 \) by a similar argument, by the induction hypothesis, we have

\[
\tau'(T_xT_i) - \tau'(T_iT_x) = \tau'(T_xT_i) - \tau'(T_iT_{x'}) + \tau'(T_IT) = \tau'(TT_i) = 0.
\]

This finishes case 1.

Case \( i = 0 \): Again, write

\[
T_x = L_{1,a_1} \cdots L_{n,a_n} T_w.
\]

Then, \( T_0T_x \in X_\Delta \) unless \( a_1 = m - 1 \). This gives us

\[
\tau'(T_0T_x) \neq 0 \iff T_x = T_0^{m-1}
\]

and in the nonzero case, there is obvious symmetry. So now, suppose \( T_x \neq T_0^{m-1} \). We want to show that \( \tau'(T_xT_0) = 0 \).

In \( S_n \), \( w \) is braid equivalent to \( s_k \cdots s_1w' \) for some \( w' \) that commutes with \( s_0 \). Hence, since the braid relations still hold in \( H_n \)

\[
T_xT_0 = L_{1,a_0} \cdots L_{n,a_n} T_k \cdots T_1 T_0 T_{w'}
\]

which has trace \( \tau' = 0 \) if \( T_{w'} \neq 1 \) by the induction hypothesis. So, assume \( T_{w'} = 1 \). Then,

\[
T_2T_0 = L_{1,a_0} \cdots L_{n,a_n} L_{k,1}
\]

and using Lemma 10.7 (c) again, we see that this gives us a nonzero \( T_1 \) coefficient if and only if \( T_x = T_0^{m-1} \). 

\[ \square \]

We finish this section by proving that \( \tau = \tau' \). This follows immediately from the following Lemma.

**Lemma 10.13.** Let \( i \geq 1 \) and let \( L_i \) be the Jucys-Murphy element defined in the previous sections. Then,
(a) For $1 \leq k \leq m - 1$, $L^k_i$ is an $A$-linear combination of
terms of the form

$$L_{1,c_1} \cdots L_{i,c_i} T_w$$

with $0 \leq c_j \leq m - 1$ for $j \neq i$ and $1 \leq c_i \leq k$.

(b) Let $1 \leq b_i \leq m - 1$. Then, $L_{b_1} \cdots L_{b_i}^i$ is an $A$-linear combination of

$$L_{1,c_1} \cdots L_{i,c_i} T_w$$

with $c_i \neq 0$.

Proof. First prove (a) and then (b) using induction on $i$ and Lemma 10.7 (c).

This lemma tells us that $T_e$ coefficients in the word basis and the Jucys-Murphy basis agree. Hence, $\tau = \tau'$.

**Part C - Affine Hecke Algebra Modules in the Kato Block**

This is the last section of these notes in which we construct an equivalence of categories between representations of $H_{F,n}$ of particular central character and representations of $Z_{F,n}$ of particular formal character, for which the essential tool will be the unique irreducibility of a Kato Module in its block. In this section, we fix $F$ as an algebraically closed field (not necessarily characteristic 0) and then omit the $F$ from the notation for the various algebras.

Additionally, in this section, we fix an $a \in F^\times$ and restrict ourselves to $H_{n}$-modules that are in the block with central character corresponding to $(a, \ldots, a)$ i.e. in the block in which the unique irreducible is the Kato Module $L(a^n)$, which we now denote as $K_n$. We thus make the following definitions in order to formulate the results better.

**Definition 10.14.**

1. Let $\mathfrak{m}_n$ be the maximal ideal in $Z_n$ that is the intersection of the two sided ideal of $\mathcal{P}_n$ generated by $(X_1 - a, \ldots, X_n - a)$. Let $\widehat{Z}_n$ denote the completion of $Z_n$ at this maximal ideal and let $\widehat{\mathcal{P}}_n, \widehat{H}_n$ be the completion of the respective algebras at $\mathfrak{m}_n$. Let $\widehat{H}_n^T$ be the completion of $\widehat{H}_n$ inside $\widehat{H}_n$.

2. Define $\mathcal{M}_n$ to be the category of $Z_n$-modules on which $\mathfrak{m}_n$ acts locally nilpotently, or equivalently as the category of $\widehat{Z}_n$-modules. Similarly, define $\mathcal{N}_n$ to be the category of left $H_n$-modules on which $\mathfrak{m}_n$ acts locally nilpotently, or equivalently as the category of $\widehat{H}_n$-modules.

Our goal is to prove that $\mathcal{M}_n$ and $\mathcal{N}_n$ are equivalent and, moreover, to specify a pair of functors that establish the equivalence of categories. Before we can state this precisely, we need to define the following element in $\widehat{H}_n$:

**Definition 10.15.** Let $\tau$ be either the trivial or sign character on $\widehat{H}_n^T$. Then, define the elements $c_n^\tau \in Z(\widehat{H}_n^T)$ as...
\[ c_{\text{sgn}}^n := \sum_{w \in S_n} q^{-l(w)} \tau(T_w) T_w \]

and

\[ c_1^n := \sum_{w \in S_n} T_w. \]

For each \( \tau \), \( c_\tau^n \) has some nice properties.

**Proposition 10.16.** The following hold:

1. For any \( T_w \in \mathcal{H}_n^T \), we have

\[ T_w c_\tau^n = c_\tau^n T_w = \tau(w) c_\tau^n. \]

and in particular, for \( n > 1 \),

\[ c_1^n c_{\text{sgn}}^n = 0. \]

2. For any projective \( \mathcal{H}_n^T \)-module \( M \),

\[ c_\tau^n(M) = \{ m \in M : T_w(m) = \tau(w) m \ \forall w \in S_n \}. \]

3. For any projective \( \mathcal{H}_n^T \)-module \( M \), the multiplication map

\[ c_\tau^n \mathcal{H}_n^T \otimes \mathcal{H}_n^T M \to c_\tau^n M \]

is an isomorphism.

4. For any \( \mathcal{H}_n \)-module \( M \), the canonical map

\[ c_\tau^n \mathcal{H}_n^T \otimes \mathcal{H}_n^T M \to c_\tau^n \mathcal{H}_n \otimes \mathcal{H}_n M \]

is an isomorphism of \( \mathbb{Z}_n \)-modules.

**Proof.** 1 is direct computation. 2 follows by looking at the case where \( M \) is free which follows from 1. 3 is straightforward from 2 and 4 follows from 3. \( \square \)

**Remark.** Note that in characteristic 0, the above proposition implies that the two \( c_\tau^n \) are mutually orthogonal simple central idempotents and in this case, the entire results of this section follow from the theory of central idempotents. In the case of positive characteristic, however, for large values of \( n \), \( c_\tau^n \) will actually be nilpotent and hence the results proved below are nontrivial.

We are now ready to state and prove the main theorem.

**Theorem 10.17.** Fix some \( \tau \in \{1, \text{sgn}\} \). Define the functor \( F : \mathcal{M}_n \to \mathcal{N}_n \) as

\[ M \mapsto \mathcal{H}_n c_\tau^n \otimes \mathbb{Z}_n M \]

and the functor \( G : \mathcal{N}_n \to \mathcal{M}_n \) as

\[ M \mapsto c_\tau^n \mathcal{H}_n \otimes \mathcal{H}_n M. \]

Then, \( F \) and \( G \) establish an equivalence of categories between \( \mathcal{M}_n \) and \( \mathcal{N}_n \).
Proof. We break the proof down into several steps.

Step 1: We show that $\mathcal{F}$ and $\mathcal{G}$ are exact functors. $\mathcal{F}$ is exact because, by 1 in Proposition 10.16, $\hat{H}_n c_n^\tau$ is free of rank 1 over $\mathcal{P}_n$ and is hence free of rank $n!$ over $\mathbb{Z}_n$ and hence is flat over $\mathbb{Z}_n$. Now, $\mathcal{G}$ is clearly right exact. To show that $\mathcal{G}$ is left exact, note that every module in $\mathcal{N}_n$ has a filtration by the Kato module $K_n$, as it is the only irreducible in the category. But $K_n$ is free over $\mathcal{H}_n^T$ and hence every module in $\mathcal{N}_n$ is projective over $\mathcal{H}_n^T$. Thus, by Proposition 10.16, the functor $\mathcal{G}$ is isomorphic to the functor $M \mapsto \{ m \in M : h \cdot m = \tau(h)m \forall h \in \mathcal{H}_n^T \}$. This functor is clearly left exact.

Step 2: We show that $\mathcal{G}$ is right adjoint to $\mathcal{F}$. Note that $\mathcal{F}$ has an obvious right adjoint $\mathcal{F}^* : \mathcal{N}_n \to \mathcal{M}_n$ which sends $M \mapsto \text{Hom}_{\mathcal{H}_n^T}(\hat{H}_n c_n^\tau, M)$. Now, $\hat{H}_n c_n^\tau$ is isomorphic as a left module to $\hat{H}_n / I$, where by Proposition 10.16 and the Basis Theorem, $I$ is the ideal $\hat{P}_n \otimes I^T$ with $I^T$ the left ideal in $\hat{H}_n^T$ generated by $(h - \tau(h))$. Then, by the proof of left exactness of $\mathcal{G}$, we have canonical isomorphisms $\mathcal{F}^*(M) \cong M^I := \{ m \in M : Im = 0 \} = M^\tau \cong \mathcal{G}(M)$. Hence, $\mathcal{G}$ is right adjoint to $\mathcal{F}$.

Step 3: We finish the proof of the theorem by showing that the counit and unit are isomorphisms. Since the functors $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are exact it suffices to show this fact for the simple objects. In $\mathcal{N}_n$, we have a unique simple $K_n$ and in $\mathcal{M}_n$, we have the unique one dimensional simple corresponding to the central character $(a, \cdots, a)$, which we denote now by $L_n$.

Now, as $\hat{H}_n c_n^\tau = \hat{P}_n c_n^\tau$ is free of rank $n!$ over $\mathbb{Z}_n$, $\dim_F (L_n) = n! \Rightarrow F(L_n) \cong K_n$ as $F(L_n)$ must contain the Kato module and both have the same dimension. Conversely, since the action of $\mathcal{H}_n^T$ on $K_n$ is just the action of $\mathcal{H}_n^T$ on itself, we see that $\dim_F G(K_n) = \dim_F K_n^\tau = 1 \Rightarrow G(K_n) \cong L_n$. Thus, the unit and counit are both nonzero morphisms between the same simple objects and are hence isomorphisms.

\[ \square \]

Remark. There is an alternative approach to proving the main theorem. We state this approach here but do not carry it through. To show that $\mathcal{F}$ and $\mathcal{G}$ establish an equivalence of categories, it suffices to prove that

$$c_n^\tau \otimes_{\mathcal{H}_n} \hat{H}_n c_n^\tau \cong \mathbb{Z}_n$$
as a left $\hat{\mathbb{Z}}_n$ module and

$$\hat{\mathcal{H}}_n c_n^\tau \otimes_{\mathbb{Z}_n} c_n^\tau \hat{\mathcal{H}}_n \cong \hat{\mathcal{H}}_n$$

as a left $\hat{\mathcal{H}}_n$ module. This is the approach taken in [CR04].

Appendix

Our construction of the irreducible representations of $H_n$ is via a deformation of the Specht module construction for $W_n$. We first define standard Young tableaux, content and axial distances for $m$-partitions of $n$.

**Definition 10.18.** Let $\lambda$ be an $m$-partition of $n$. Then, a standard Young tableau for the $l$-tableau associated to $\lambda$ is a filling in of the boxes of the $m$-tableau with the numbers $1, \ldots, n$ such that in each component tableau, the enumeration is standard.

Let $a, b$ now be boxes in $\lambda$. Then, we define the content $c(\lambda; a)$ of $a$ to be the row index of $a$ minus the column index of $a$, in the component tableau in which $a$ lives. Additionally, we define the axial distance

$$r(a, b) = c(\lambda, a) - c(\lambda, b).$$

Finally, for an integer $l$ and an indeterminate $y$, we first define

$$\Delta(l, y) = 1 - q^l y \in \mathbb{A}[y]$$

and then define the following $2 \times 2$ matrix

$$M(l, y) = \frac{1}{\Delta(l, y)} \begin{pmatrix} q - 1 & \Delta(l + 1, y) \\ q\Delta(l - 1, y) & -q^l y(q - 1) \end{pmatrix}.$$

Now, let $\lambda \vdash_m n$ and let $V_\lambda$ be the free $K$-vector space with basis the set of standard Young tableaux of shape $\lambda$. For any $i \in \{1, \ldots, n\}$ and for any standard Young tableau $M_\rho$ of shape $\lambda$, define $\tau_\lambda(i)$ to be the index of the component tableau of $\rho$ that $i$ appears in. We now define a representation of $H_n$ on $V_\lambda$ as follows:

1. $T_0 M_\rho = q_{\tau_\rho(i)} l_\rho$.

2. For $i > 0$, we have 3 cases for the action of $T_i$ on $M_\rho$:
   a. If $i, i + 1$ lie in the same row of the same component diagram of $M_\rho$, then $T_i M_\rho = q M_\rho$.
   b. If $i, i + 1$ lie in the same column of the same component diagram of $M_\rho$, then $T_i M_\rho = -1 M_\rho$.
   c. If neither of the above hold, let $M_\rho'$ be the standard Young tableau with $i$ and $i + 1$ swapped. Then,

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\[ T_i \langle M_\rho, M_{\rho'} \rangle = \langle M_\rho, M_{\rho'} \rangle M \left( r(i + 1, i), \frac{q_{\tau_\rho(i)}}{q_{\tau_\rho(i+1)}} \right). \]

This is well-defined because if \( i, i + 1 \) lie in the same component diagram, then the axial distance between them cannot be \( 0, -1 \) unless they are in the same row or column.

As an example, and also because it will be useful in proof of irreducibility, we compute the matrix corresponding to the action of \( T_{n-1} \) in case 2(c). We let \( d \) denote \( r(n, n-1) \), \( a \) denote \( \tau_\rho(n-1) \) and \( b \) denote \( \tau_\rho(n) \). Then, the action of \( T_{n-1} \) is given by the matrix

\[
M \left( r(n, n-1), \frac{q_{\tau_\rho(n-1)}}{q_{\tau_\rho(n)}} \right) = \frac{1}{1 - q^{d-1} \frac{q_a}{q_b}} \begin{pmatrix} q - 1 & \frac{1}{1 - q^{d+1} \frac{q_a}{q_b}} \\ \frac{q}{1 - q^{d+1} \frac{q_a}{q_b}} & -q^{d}(q - 1) \end{pmatrix}.
\]

We denote this special matrix by \( N \), and note that, since \( q \) is a unit, \( N_{21} \) is nonzero.

References


