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Stretchability of Pseudolines is NP-Hard

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ABSTRACT. We prove that the problem of determining whether a pseudoline arrangement is stretchable is NP-hard. We also use our techniques to find a symmetrical pseudoline arrangement that is stretchable but not stretchable to a symmetrical line arrangement. The NP-hardness result can also be obtained from a paper by Mnëv (Lecture Notes in Math., vol. 1346, Springer, 1988, pp. 527–544) which implies the stronger result that determining stretchability is equivalent to the existential theory of the reals. We give a short explanation of Mnëv's proof, viewed from a complexity theory point of view, which may be more comprehensible than the original paper to readers who do not know much topology.

1. Introduction

A *line arrangement* is the partition of the plane induced by a set of lines in the plane. A *pseudoline* is a simple curve in the plane that goes to infinity in two directions. (In other words, a pseudoline is the image of a line under a homeomorphism of the plane.) A *collection of pseudolines* is a set of pseudolines such that any two members of the set intersect at most once, and cross if they intersect. A *pseudoline arrangement* is the partition of the plane induced by a collection of pseudolines. A pseudoline arrangement is *stretchable* (or *realizable*) if there is an arrangement of lines with the same combinatorial structure. A line or pseudoline arrangement is *uniform* if no three lines intersect in a point and no two lines are parallel.

The stretchability of pseudoline arrangements has a long history. In particular, given a pseudoline arrangement, the question of finding a realization of it has been studied extensively [**BS**]. We show that this problem is NP-hard. We also answer a question of Bokowski and Sturmfels [**BS**, p. 80] by showing that there exists a symmetrical pseudoline arrangement which is stretchable, but which is not stretchable to a symmetrical line arrangement.

The *realization space* of a pseudoline arrangement is the set of line arrangements realizing this pseudoline arrangement. Mnëv [**Mn**] has shown

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that the topology of the realization space of a pseudoline arrangement can be the same as the topology of any semialgebraic variety (a *semialgebraic variety* is the solution space of a set of polynomial inequalities and equations over the reals). Mnëv's result also implies that determining the stretchability of a pseudoline arrangement is equivalent to the existential theory of the reals. This is stronger than our result.

In §2 of this paper we give our proof of the NP-hardness of determining if a pseudoline arrangement is stretchable. In §3, we use our techniques to show that there exists a symmetric pseudoline arrangement which is stretchable, but not stretchable to a symmetric line arrangement. In §4, we give Mnëv's proof that determining the stretchability of a pseudoline arrangement is equivalent to the existential theory of the reals, with one argument in his proof simplified so as to require less topology than he uses. In §2 and 3, we will be working with pseudoline arrangements—our basic objects will be lines. In §4, we will (as Mnëv does) work with "pseudo-point" arrangements, i.e., a configuration of points for which we know the orientation of all triples. This is an equivalent problem, as pseudo-point arrangements are the projective dual of pseudoline arrangements.

2. Proof of NP-hardness

Our proof is based on incidence theorems of projective geometry, namely Pappus' and Desargues' theorems. More specifically, we use the nonrealizable arrangements of pseudolines that can be obtained from these two theorems. The Pappus and Desargues configurations are shown in Figures 1 and 2. The Pappus configuration contains nine lines, each incident with three of the points, and nine points, each on three of the lines. In the Desargues configuration, there are ten lines and ten points. (Note that we do not draw the Desargues configuration in the standard manner.) Pappus' theorem is that in the Pappus configuration, if any eight of these triples of lines are concurrent (or eight of the triples of points are collinear), the last triple must also be concurrent (collinear). Similarly, Desargues' theorem is that if nine of these triples of lines (points) are concurrent (collinear), the last triple must also be. From either of these configurations, we can obtain a nonrealizable arrangement of pseudolines. We do this by slightly bending each of the lines to "go around" the points; that is, we replace each of the points in the configuration by a small triangle. By bending all lines in the right way (see Figure 3 and 4), we obtain nonrealizable uniform configurations [**Gr**].

We prove NP-completeness by reducing a variant of the NP-complete 3-SAT problem to the stretchability problem. The 3-SAT problem is: Given a Boolean expression in conjunctive normal form containing only three variables in each clause, is there an assignment of the variables making the expression true [**GJ**]? It is easy to show that this problem is still NP-complete if we require that in each clause either only nonnegated variables or only negated variables appear. We use this variant of 3-SAT, called *monotone* 3-SAT

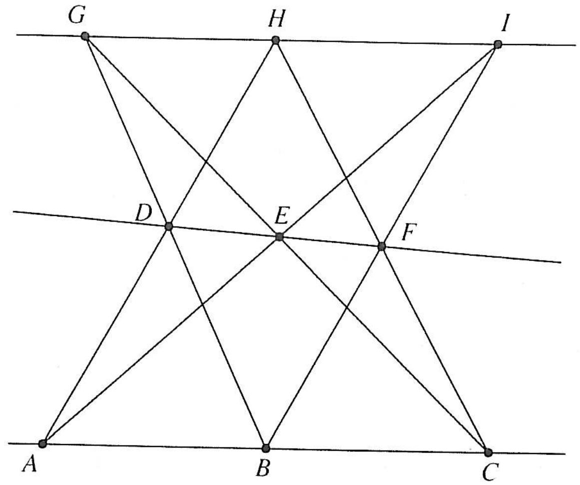

FIGURE 1. A Pappus configuration.

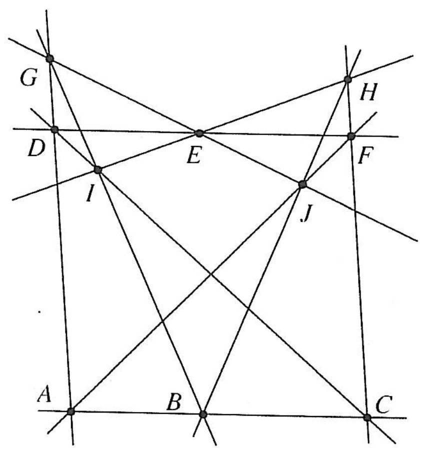

FIGURE 2. A Desargues configuration.

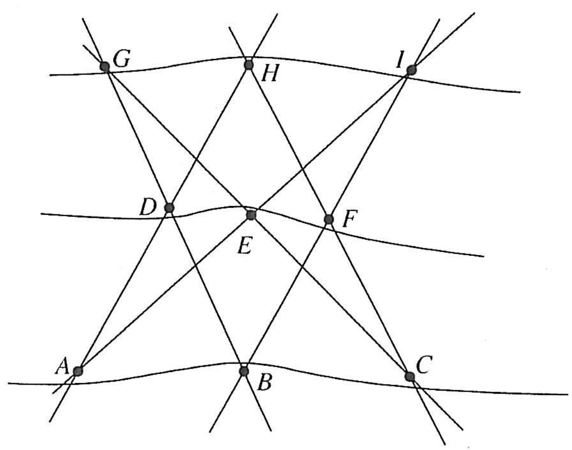

FIGURE 3. A nonrealizable Pappus configuration.

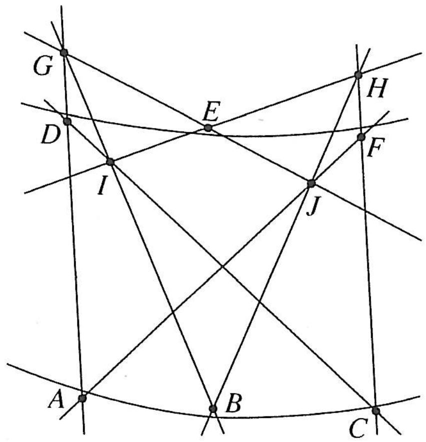

FIGURE 4. A nonrealizable Desargues configuration.

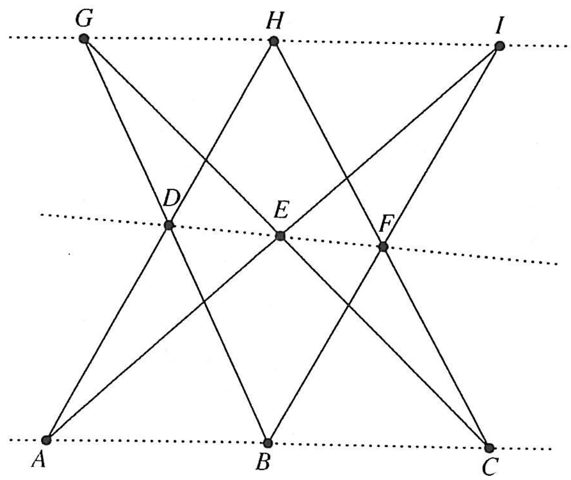

FIGURE 5. A Pappus configuration with three "imaginary" lines.

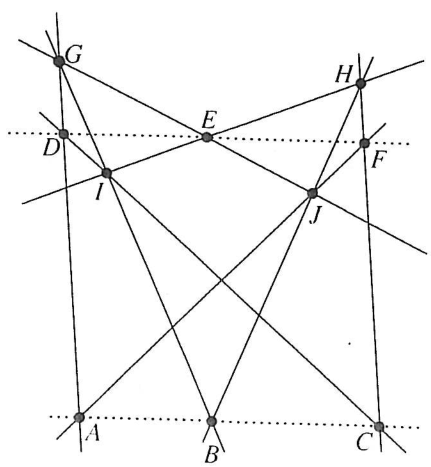

FIGURE 6. A Desargues configuration with two "imaginary" lines.

[GJ, p. 259]. We call a clause *positive* if it contains only nonnegated variables and *negative* if it contains only negated variables. To show monotone 3-SAT is NP-hard, we can reduce 3-SAT to it by adding a separate variable y_i for each variable x_i, and setting $x_i = \overline{y_i}$ by adding the clauses $x_i \vee y_i \vee 0$ and $\overline{x_i} \vee \overline{y_i} \vee \overline{1}$. We then replace $\overline{x_i}$ by y_i where needed so as to make all clauses either positive or negative.

Given a monotone 3-SAT formula, we will construct a pseudoline arrangement which is stretchable if and only if the formula is satisfiable. In the pseudoline arrangement, clauses will correspond to modified Pappus configurations, variables will correspond to certain triples of points, and variables will be linked to clauses by modified Desargues configurations.

We now look at the Pappus configuration more closely. We let three of the lines be "imaginary" (see Figure 5), i.e., the lines do not appear in our final pseudoline arrangement, but are used implicitly in the proof. We call the three points on each of these "imaginary" lines a *triple*. In our construction, each Pappus configuration corresponds to a clause of the Boolean formula and each of the three triples in the Pappus configuration corresponds to one of the variables in the clause. Furthermore, the position of the points in the triples corresponds to the truth (or falsehood) of the variables. Specifically, let PQR be the triple corresponding to the variable x_i, with point Q between points P and R. We will put point Q above (or on) the line PR if x_i is true and below if x_i is false. Unless all three triples of points are collinear, the Pappus configuration is realizable if and only if not all the variables are the same; i.e., if in at least one of three triples ABC, DEF, and GHI the middle point is above the segment between the end points of that triple, and in at least one of these triples the middle point is below the segment.

In our pseudoline arrangement, we will have many different Pappus configurations, one corresponding to each clause. This gives rise to many triples all corresponding to the same variable. We must connect these triples somehow so as to ensure that they all give the same value of x_i. To do this, we introduce three new points for the variable x_i, and make the position of these three points correspond to the value of x_i. We will call these three points the *master triple* for x_i. We then hook this master triple up to all the *slave triples* that correspond to x_i in the Pappus configurations. We could do this by using more Pappus configurations, but for reasons which we will explain later we use Desargues configurations instead. In a Desargues configuration, if we take out the two lines ABC and DEF (see Figure 6), the resulting configuration is realizable only if point B is above line AC and point E is below line DF or if point B is below and point E is above. Assume we are dealing with a positive clause. We wish to hook up one of the slave triples in the clause (corresponding to variable x_i) with the master triple for x_i. To do this we introduce a new Desargues configuration with the bottom triple ABC being the master triple and the top triple DEF being the slave triple. We now define true and false for a master triple to the opposite of true and false for a slave triple; i.e., in a master triple, for a false variable, the middle point is above the other two points, and for a true variable, the middle point is below the other two points. This ensures that if the points in the master triple are false, then the points in the slave triple are false (and vice versa). Our actual construction will use a modified Desargues configuration having only the property that if the points in the master triple are false, then the points in the slave triple are. However, this is exactly what we want: in a Pappus configuration corresponding to a positive clause, the configuration is nonrealizable if and only if the three corresponding master triples are in the false position. For Pappus configurations corresponding to a negative clause, we will turn the Desargues configuration upside-down, so the Pappus configuration is nonrealizable if and only if the three master triples are in the true position.

The astute reader may have noticed that we are using both Desargues and Pappus configurations, where intuitively it seems that only one of these configurations would suffice. We could probably obtain a proof using just one of these configurations, but each has properties which make the proof simpler if we use them both. The advantage of the Desargues configuration is that it has more degrees of freedom than the Pappus configuration; this makes it easier to show that a pseudoline configuration corresponding to a satisfiable formula is stretchable. The advantage of the Pappus configuration is that there are three disjoint lines contained in a Pappus configuration, one for each of the variables in a clause, whereas a Desargues configuration contains only two disjoint lines.

We are now ready to give the construction. First, we describe how to construct the pseudoline arrangement corresponding to a given Boolean formula. This arrangement will be stretchable if and only if the formula is satisfiable. During most of the construction of our pseudoline arrangement, we will actually be constructing a line arrangement. It is only in the last step of our construction that we will perturb this line arrangement to get a pseudoline arrangement that may not be stretchable.

To start constructing the arrangement, for each variable we place a triple of three points (the master for this variable) on a line. All these variables will be placed near some horizontal line, say $y = 0$, but in "general position"; i.e., the only relations between the points are that the three points in a triple lie on a line. This can be accomplished by putting all the triples down on the line $y = 0$ and then perturbing them slightly (see Figure 7). We will later show how to accomplish this in polynomial time.

We must include the values 1 and 0 among our variables. There are two ways of doing this: we can either add an extra line in the master triples corresponding to 0 and 1, which forces the middle point to be above (or below) the line through the end points, or we can add a set of clauses which forces one variable to be 0 and another to be 1.

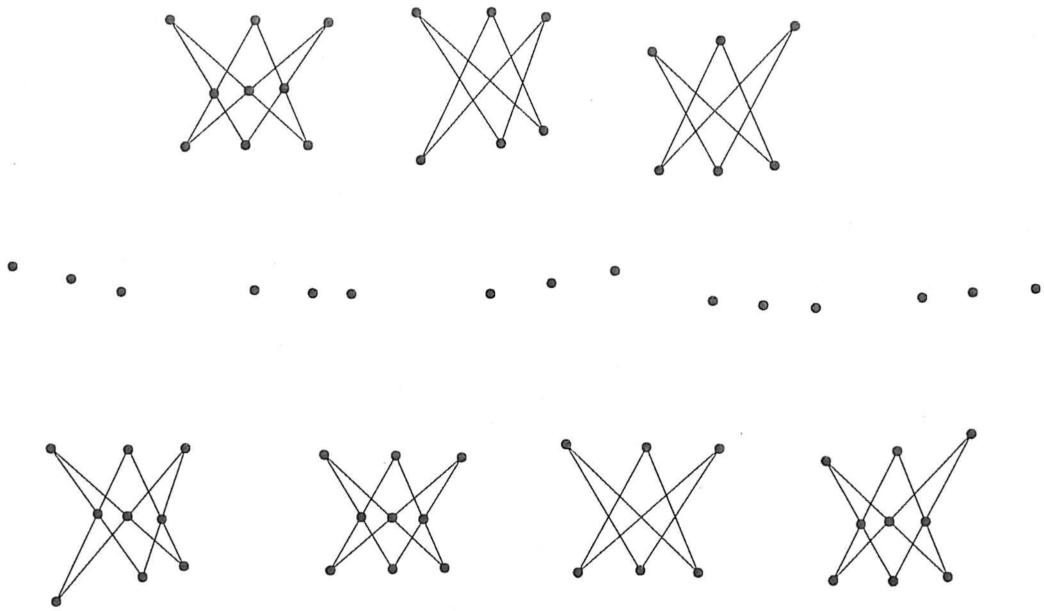

FIGURE 7. The general layout of our construction, with Pappus configurations corresponding to positive clauses on top, Pappus configurations corresponding to negative clauses on bottom, and master triples in the middle.

Next, for each positive clause, we place a modified Pappus configuration above the horizontal line $y = 0$ which all the master triples lie near, and for each negative clause we place a modified Pappus configuration below this horizontal line (see Figure 7). Here, by modified Pappus configurations, we mean a Pappus configuration missing the three horizontal lines as in Figure 5. Again, although all the triples in these modified Pappus configurations are nearly horizontal, all these configurations must lie in general position, so the only relations between points are those implied by the fact that they are all modified Pappus configurations. The important part of this placement is that the top side of each of the master triples "sees" the bottom of the triples in positive Pappus configurations, and the bottom of each of the master triples "sees" the top of the triples in negative Pappus configurations.

Next, with a "contracted" Desargues configuration (see Figure 8(a)), we will connect each master variable to the Pappus configurations corresponding to clauses containing it (see Figure 9(a)). A *contracted Desargues configuration* has had points D, G, and I identified and points F, H, and J identified, so that it contains only six points, as in Figure 8(a).

We have now placed down essentially all the lines that we need in our arrangement. We will obtain the final arrangement by slightly perturbing the arrangement that we already have. In our perturbation, we will replace certain lines by pairs of lines which differ by a small angle. Also, in the neighborhood of certain points which have many lines passing through them, we will perturb the lines so that they no longer are all concurrent. We can perturb the lines to make only the desired changes because everything is in general position.

We perturb the configuration by replacing each of the contracted Desargues configurations by a new, more complete Desargues configuration. Specifically, we replace the configuration in Figure 8(a), first by the configuration in Figure 8(b), and then by the one in Figure 8(c). Thus, we perturb the two lines b_1 (originally BD) and b_2 (originally BF) slightly inward, and add the new points $G = b_1 \cap AD$, $H = b_2 \cap CF$, $I = b_1 \cap CD$, and $J = b_2 \cap AF$. Next, line DEF is replaced by the two lines GEJ and IEH. This gives the configuration in Figure 8(b). Figure 9(b) shows this configuration connected to a Pappus configuration as it would appear in our construction.

As the points (and Pappus configurations) were placed in general position, the only triple intersections containing lines b_1, b_2 or DF in the original Desargues configuration are B, D, E, and F. If the amount we perturb the lines b_1 and b_2 by is sufficiently small, the only place that the arrangement changes is in the neighborhood of D, E, and F. It is easy enough to see what happens at D and F (see blown-up neighborhood of D in Figure 10); all lines previously passing through E still pass through it.

So far, it is possible to make all our perturbations using not just pseudoline arrangements but actual line arrangements, as in Figure 9(b). The arrangement can still be realized with every triple of points lying on a line and with all our incomplete Pappus and Desargues configurations being actual Pappus and Desargues configurations and not perturbed versions of them. This all changes in the next step.

The last step in our construction is to replace the points I and J by very small triangles $I_1I_2I_3$ and $J_1J_2J_3$, as shown in Figures 8(c) and 9(c). This is easy to do with pseudolines. In a line arrangement, by Desargues' theorem, replacing I and J by triangles in this way either forces point E or point B to move down. This makes it no longer possible to realize the pseudoline arrangement while having every triple of points lie on a line, and thus forces decisions about how to perturb these triples. We will show that it is possible to find a consistent set of such decisions if and only if the Boolean formula is satisfiable.

We will show that the above procedure finds, given a monotone 3-SAT Boolean formula, a pseudoline arrangement which is equivalent. That is, the pseudoline arrangement is realizable if and only if the expression is satisfiable. We still have not quite shown how to produce the pseudoline arrangement in polynomial time—we must give an algorithm for our initial step of putting the Pappus configurations and the sets of three points in a row down in "general position." As one might expect, this step is not hard. We also have not yet obtained a *uniform* pseudoline configuration, which we must do to show that the problem is NP-complete in the uniform case. There is a trick discovered by both Mnëv and Sturmfels and White [Mn, SW] which we can apply to turn the above configuration into a uniform one. We will discuss these steps in more detail later.

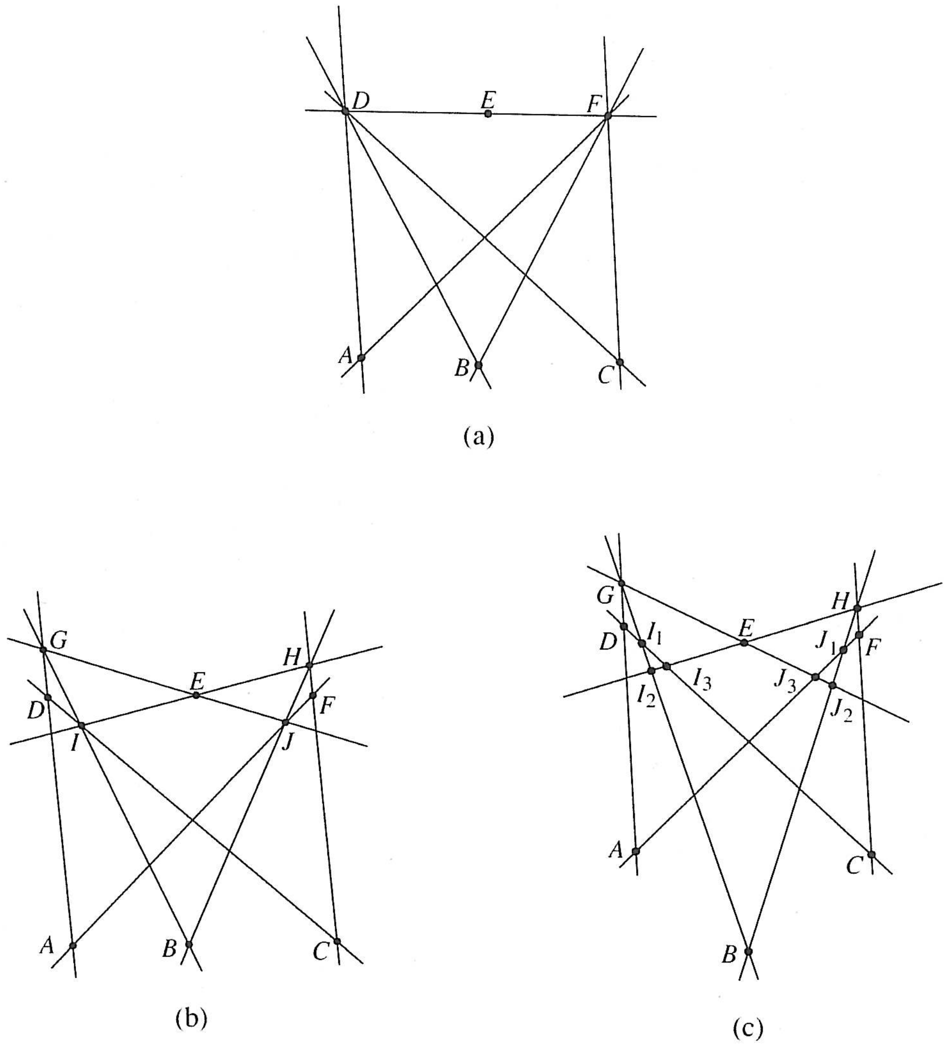

FIGURE 8(a) A "contracted" Desargues configuration. (b) The intermediate step in expanding a contracted Desargues configuration. (c) The final step: replacing points I and J with triangles.

What we do now is to show that the above pseudoline configuration is stretchable if and only if the expression generating it is satisfiable. We first do the easy direction: if the configuration is stretchable, then the expression is satisfiable. We later do the hard direction.

Suppose the arrangement is stretchable. Consider a realization of the arrangement. Let the value of a variable x_i be true if, in the master triple corresponding to x_i, the middle point is below the line segment joining the other two points. Let the variable be false if the middle point is above this segment. If the point is on the segment, we may let the variable be either false or true. Now if this truth assignment makes the Boolean expression

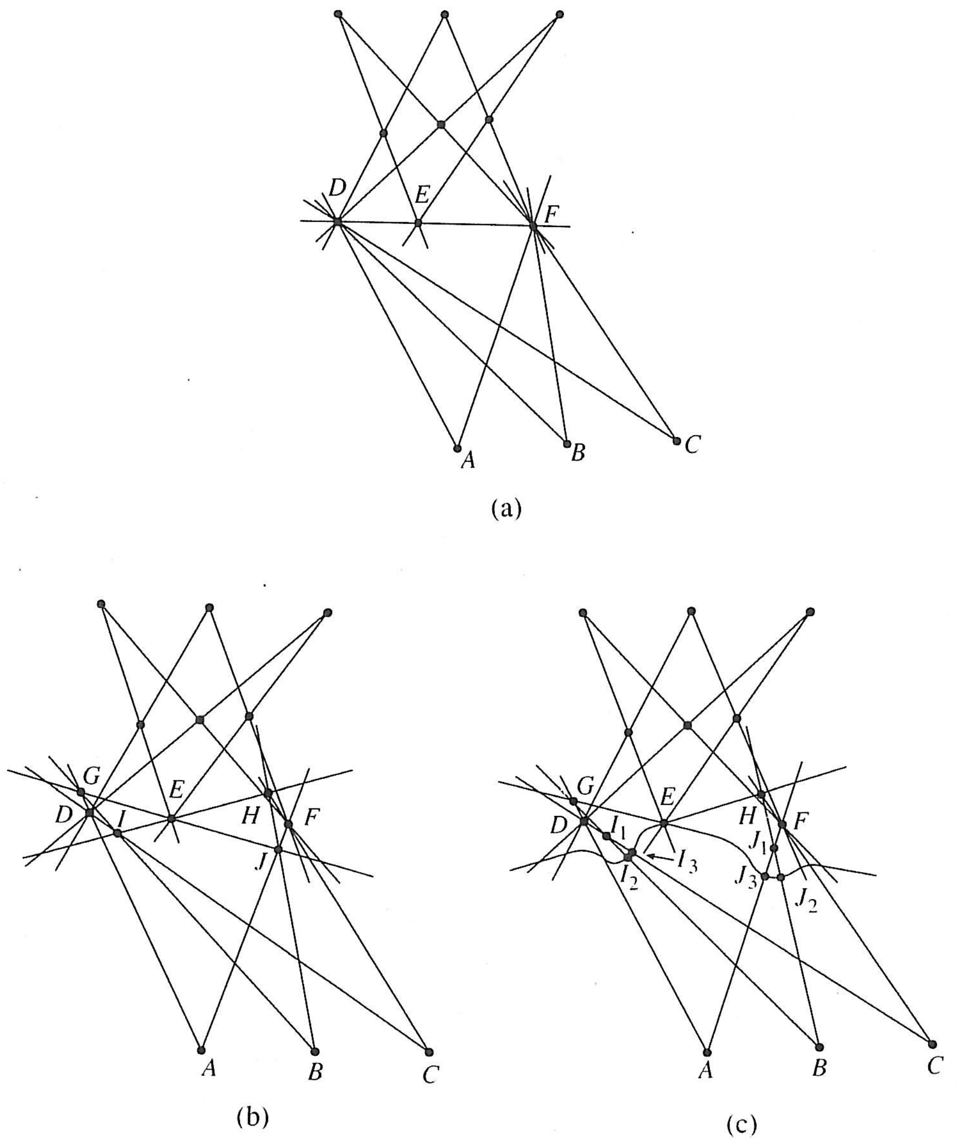

FIGURE 9(a) A contracted Desargues configuration in place: ABC is the master triple and DEF the corresponding slave triple in the Pappus configuration at top. (b) The intermediate step in expanding the contracted Desargues configuration in (a). (c) Replacing points I and J in (b) with triangles.

false, there must be some clause which is false. If this clause is a positive clause, say $x_{i_1} \vee x_{i_2} \vee x_{i_2}$, then all the variables in it are false. This means that, in the master triple corresponding to each of these variables, the middle point is above (or on) the line segment joining the other two. Since the points in this master triple are connected by a modified Desargues configuration to the points in the slave triples in the Pappus configuration corresponding to this clause, the middle point lies strictly below the line segment joining the other two points in all three triples in this Pappus configuration. However,

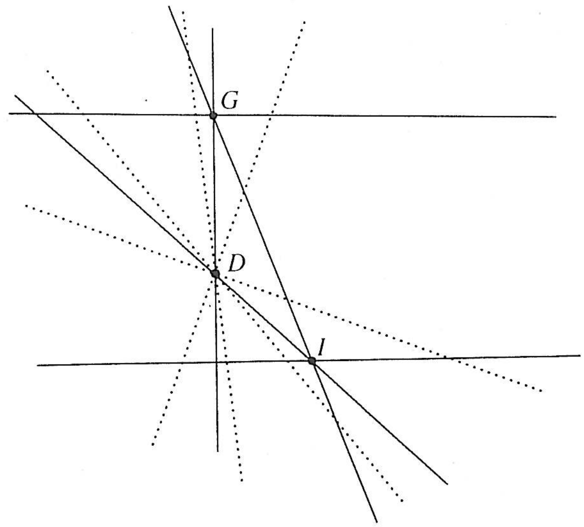

FIGURE 10. The neighborhood of point D before point I is replaced by a triangle.

this gives a nonstretchable version of the Pappus configuration. Similarly, a false negative clause also gives a nonstretchable configuration.

The other direction is harder. Assume that we are given a satisfying assignment of the Boolean formula. We must show that the corresponding pseudoline arrangement is stretchable. We start by putting the Pappus configuration and the master triples down in general position, with the three points in all triples collinear. We will then show that we can move the middle points of the master triples up or down, depending on whether the corresponding variable is false or true, perturb the Pappus configurations by a very small amount, and add the Desargues configurations linking the slave triples to the "master" triples to give a realization of the desired pseudoline configuration. We will do this by proving a series of lemmas showing that we can realize the configuration by perturbing the original configuration in a certain way.

Before we can proceed with the lemmas, we must give some definitions. For each line in our configuration, we will call two or three points on it "anchor points." We will consider the line to be "fixed" to these points, so that when these points are perturbed, the line is perturbed. For the Desargues configurations, the lines, listed by their two anchor points, are AD, AF, CD, CF, BG, BH, EG, EH. For the Pappus configurations, the anchor points will be ADH, AEI, BDG, BFI, CEG, and CFH; when we move anchor points in a line having three anchor points, we must make sure that they remain collinear. We use anchor points to bound the effects of small perturbations on the overall configuration.

LEMMA 1. *There exists an ε such that if all anchor points are moved by at most ε, any three lines not originally concurrent retain their relative orientations.*

PROOF. This is clear for any specific three lines. To obtain an ε that works for all lines, simply choose ε to be the minimum over all sets of three lines of the ε's for each set. □

For our next lemma, when we talk about the line arrangement of a Desargues configuration, we also include all lines passing through points in the configuration, although these lines may technically not be in the configuration.

Using this lemma, we will show that there is some ε_1 and a corresponding ε_2 such that if we move all the middle points of the master triples either up or down by ε_1, depending on whether the corresponding variable in the satisfying assignment is true or false, then the points of the Pappus configurations can be moved by a distance ε_2 so as to realize the arrangement. We first show the existence of these ε's for each Desargues configuration separately, and then show that we can choose these ε's to be valid for the whole arrangement. Showing that these ε's exist for each Desargues configuration is the substance of Lemma 2.

LEMMA 2. *For each Desargues configuration connecting a master triple to a positive clause, the following assertions hold*:

(1) *For every ε_1, $0 < \varepsilon_1 \le \varepsilon/2$, there exists an $\varepsilon_2 > 0$ such that if point B is moved up by ε_1 to B', if points D, E, and F are moved by less than ε_2 to D', E', and F', and if point E' is below line $D'F'$, then the points G, H, I_1, I_2, I_3, J_1, J_2, and J_3 can be added so as to make the resulting line arrangement of the Desargues configuration be combinatorially the desired arrangement, even if all the anchor points not in this Desargues configuration are moved by at most $\varepsilon/2$.*

(2) *For every ε_1, $0 < \varepsilon_1 \le \varepsilon/2$, there exists an $\varepsilon_2 > 0$ such that if point B is moved down by ε_1 to B', if points D, E, and F are moved by less than ε_2 to D', E', and F', and if point E' is above line $D'F'$, then the points G, H, I_1, I_2, I_3, J_1, J_2, and J_3 can be added so as to make the resulting line arrangement of the Desargues configuration be combinatorially the desired arrangement, even if all the anchor points not in this Desargues configuration are moved by at most $\varepsilon/2$.*

(3) *There exists an $\varepsilon_1 > 0$ such that if point B is moved down by at most ε_1 to B', then the points G, H, I_1, I_2, I_3, J_1, J_2, J_3 can be added so as to make the resulting line arrangement of the Desargues configuration be combinatorially the desired arrangement, even if all the anchor points not in this Desargues configuration are moved by at most $\varepsilon/2$.*

PROOF. (1) We will first show (1), assuming that I_1, I_2, and I_3 are all identical to point I, and similarly the J_i's are identical to J, as in Figure 8(b). Let us consider points D' and F' to be fixed. Let us also consider all anchor points outside the Desargues configuration to be fixed (except on lines with three anchor points, one of them a point in the Desargues configuration, in which case we only fix one of the two outside anchor points). As B was moved upward to produce B', by Desargues' theorem no matter where we place the lines BG and BH, E' will be below $D'F'$. However, we can make E' approach segment $D'F'$ by moving line BG towards D' and BH towards F'. By moving only BG towards D', we move E' to the right, and by moving only BH towards F', we move E' to the left. We can therefore move these two lines simultaneously so as to make E' approach any point on segment $D'F'$. Thus, in some sufficiently small neighborhood of E, we can put E' anywhere below $D'F'$. This involves moving points G, H, I, and J by some amount that we can make arbitrarily small by making ε_2 arbitrarily small. It is easy to see that by making ε_2 sufficiently small, the combinatorial structure of the arrangement near points B, D, E, and F is unaffected. By Lemma 1, the combinatorial structure cannot be affected anywhere else. Finally, we can perturb the lines GE and HE by a tiny amount to make triangles at I and J without affecting the combinatorial structure of the arrangement elsewhere.

We have now proved (1) with points D', F', and anchor points outside the Desargues configuration all fixed. However, by compactness, this implies (1) even when these points are not fixed: the set of all positions for these points at most $\varepsilon/2$ away from their original positions is a compact set, and as the maximum possible ε_2 is a continuous function of these positions, ε_2 as a function of the positions of these points must be bounded away from 0 on this set.

(2) This proof is very similar to the proof of (1), and will be omitted.

(3) We first place points G and H on lines AD and CF, sufficiently close to points D and F, respectively, so that the combinatorial configuration is as in Figure 8(b), even if all the other anchor points are moved by at most $\varepsilon/2$. Now, for the time being, let us fix all these other anchor points. Suppose B' is below line AC. Let B'' be the intersection of lines AC and $B'E$. Since B'' is on line AC and E is on line DF, by moving point G closer to D or point H closer to F, we can keep point E fixed and move point B to B''. Now, note that if we move B'' downward along line $B'E$ to point B', keeping points G and H fixed, we open small triangles at points I and J, producing the desired configuration of Figure 8(c). The amount that we can move B'' downward is constrained by what happens to other lines through D and F (see Figure 11). For example, if we move B'' down too far, we may put I_2 on the wrong side of a different line through point D (to another master triple). However, if we keep points B'' and B' close enough to B so as to move lines BG and BH through a sufficiently small angle, we do not change the combinatorial configuration by moving lines BG or BH. Thus, if we make ε_1 small enough, the angle $B'GB$ becomes arbitrarily small, and we obtain the results in (3). Again, by compactness, the proof with anchor points fixed implies the result with nonfixed anchor points. □

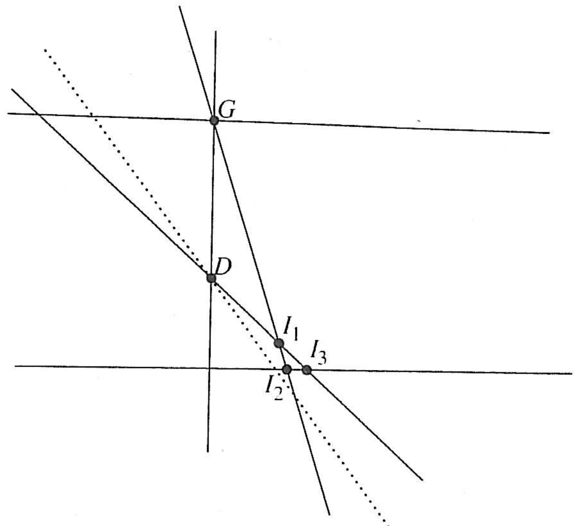

FIGURE 11. The dotted line must pass below point I_2.

Now, we use Lemma 2 to prove the following:

LEMMA 3. *In our construction, if the Boolean formula is satisfiable then the corresponding pseudoline arrangement is realizable.*

PROOF. To show the pseudoline arrangement is realizable, we use Lemma 2. Assume we are given a satisfying assignment for the Boolean formula. Consider a Pappus configuration in our construction. This configuration corresponds to a clause of the Boolean formula. Assume that this is a positive clause, so the Pappus configuration is above the line $y = 0$. (If it corresponds to a negative clause, the argument is symmetric.) In the satisfying assignment, this clause must have either one, two, or three variables set to true, and the rest set to false. If all three variables are set to true, we do not move any points of the Pappus configuration. If two variables are true and one false or if two variables are false and one true, we move all the points of the Pappus configuration by at most ε_2 (which is a quantity that will be determined later) so that for the true slave variables, the middle point is above the other two, and for the false slave variables, the middle point is below the other two. (This can easily be done.) Now, we go back to Lemma 2. We first choose an ε_1 smaller than all the ε_1's of Lemma 2(3) for the Desargues configurations in our construction. We then move all middle points of the master triples up or down by exactly ε_1, depending on whether the variable is false or true. We then choose ε_2 so it is smaller than all the ε_2's of Lemma 2(1), (2) for the Desargues configurations in our construction. Now, by Lemma 2, each Desargues configuration can be perturbed appropriately, and by Lemmas 1 and 2 none of these perturbations interferes with another (all the anchor points have been moved by less than $\varepsilon/2$), so we have a realization of our pseudoline arrangement. □

To conclude the proof that determining stretchability of pseudolines is NP-hard, we need to show how to construct the pseudoline arrangement in polynomial time. To do this, we need to show how to lay out the master triples and the Pappus configurations in general position. One way to do this is to first lay out all the configurations, but not necessarily in general position, and then perturb them to obtain a configuration in general position. We do not need to obtain actual coordinates for the perturbed lines; we merely need the combinatorial structure of the resulting line arrangement. It thus suffices to perturb the lines symbolically. To perturb a Pappus configuration, we first symbolically translate the configuration by an "infinitesimal" amount. This eliminates all degeneracies between lines in the Pappus configuration and lines outside it, except for those at points at infinity. To remove these, we symbolically rotate the Pappus configuration. A similar procedure works to put the master triples into general position.

We have now shown that determining the stretchability of a pseudoline arrangement is NP-hard. The arrangements we have constructed, though, are not uniform. To show that it is still NP-hard with a uniform arrangement, we use a lemma proved both by Mnëv and by Strumfels and White on *constructible* pseudoline arrangements.

A pseudoline arrangement is *constructible* if we can produce it by adding the pseudolines one at a time, while never placing a pseudoline through more than two points defined by intersections of previously placed pseudolines.

LEMMA 4 [**Mn, SW**]. *Given a constructible pseudoline arrangement $\mathscr{L}$ one can find (in polynomial time) a uniform pseudoline arrangement $\mathscr{L}'$ which is stretchable if and only if $\mathscr{L}$ is stretchable.*

PROOF. To produce $\mathscr{L}'$, consider the order of placing the pseudolines that shows the arrangement $\mathscr{L}$ is constructible. We will examine the pseudolines in the reverse order, and replace pseudolines that passed through one or two points previously defined by intersections by either two or four new pseudolines. When processing a pseudoline L that passes through one point P defined by intersections of pseudolines, replace it by two pseudolines as in Figure 12. When processing a pseudoline L that passes through two points defined by intersections of pseudolines, replace it by four pseudolines as in Figure 13. This process will produce a uniform pseudoline arrangement $\mathscr{L}'$. Clearly, if $\mathscr{L}$ is stretchable, $\mathscr{L}'$ is. If $\mathscr{L}'$ is stretchable, one can produce a stretching of $\mathscr{L}$ by starting with the realization of $\mathscr{L}'$ and undoing the process we used to construct $\mathscr{L}'$, as shown in more detail in [**Mn, SW**]. □

Now, to show the NP-hardness result applies to uniform arrangements, we need to show that the arrangement we produced is constructible. If we first put down the master triples and the Pappus configurations, and then add

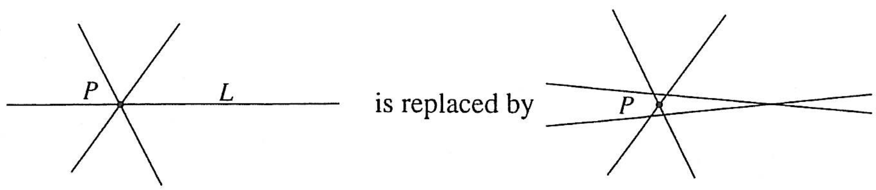

FIGURE 12. Making a constructible arrangement uniform: replacing a line through one point.

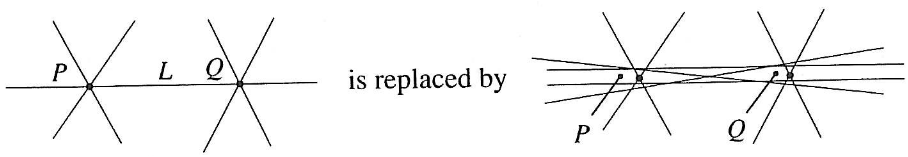

FIGURE 13. Making a constructible arrangement uniform: replacing a line through two points.

the lines in each of the Desargues configurations in the order ADG, CFH, EH, EG, CD, AF, BG, BH, we never put a line through more than two previously defined points. Our arrangement is thus constructible, so by Lemma 4 realizability of uniform pseudoline arrangements is also NP-hard.

3. A symmetric pseudoline arrangement not symmetrically stretchable

By using the tools introduced in the previous section, it is easy to produce a symmetrical pseudoline arrangement that is not stretchable to a symmetrical line arrangement. We will construct this in much the same manner that we constructed the pseudoline arrangement from an arbitrary 3-SAT formula in the previous section. This arrangement will correspond to the formula $(x \vee y) \wedge (\bar{x} \vee \bar{y})$. The only satisfying assignments for this formula are, clearly, $x = 1$, $y = 0$, and $x = 0$, $y = 1$. We will construct the symmetric pseudoline arrangement with a vertical axis of symmetry. To do this, we put down on the left of the vertical axis, in general position, a master triple for x, and on the right of the vertical axis, symmetrical to the triple for x, a master triple for y. On the left of the axis of symmetry, we put in general position a Pappus configuration both above and below the master triples. We then put these Pappus configurations symmetrically to the right of the line (see Figure 14). Since these Pappus configurations correspond to clauses containing two variables, we add an extra line through one of the three triples (say, through points DEF in Figure 5). Now, using Desargues configurations as in the previous section, we connect the master triple for x to points ABC in the Pappus configurations on the left and to points GHI

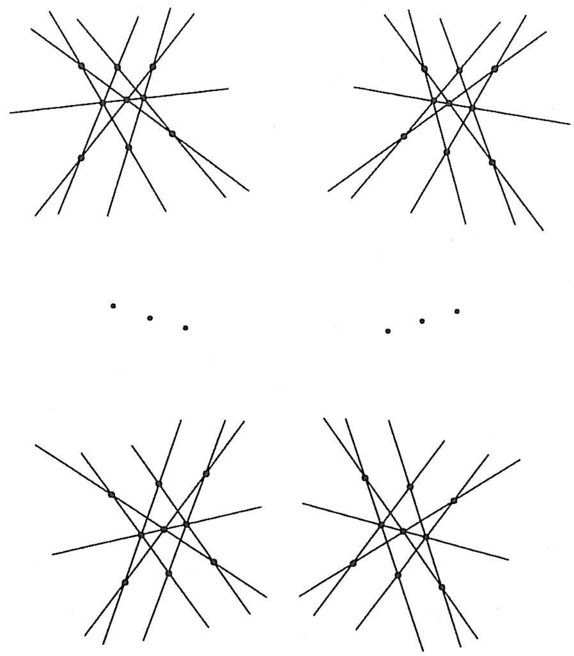

FIGURE 14. The master triples and Pappus configurations for a symmetrical pseudoline arrangement not symmetrically stretchable.

in the Pappus configurations on the right. Similarly, we connect the master triple for y to points GHI in the Pappus configurations on the left and to points ABC in the Pappus configurations on the right. By the arguments in the previous section, the resulting pseudoline configuration is realizable only by configurations in which the middle point of the master triple for x is above the line through the other two points in this triple and the middle point of the master triple for y is below the line through the other two points, or vice versa. Again, by the arguments in the previous section, this configuration can indeed be realized. Thus, we have a symmetric pseudoline arrangement which is not symmetrically stretchable.

By using Lemma 4, we can turn this pseudoline arrangement into a uniform one. All we need to show is that we can apply Lemma 4, and retain the symmetry. We can do this because in this arrangement the intersection of a symmetric pair of lines never lies on more than these two lines.

4. A computer scientist's view of Mnëv's universality theorem

Mnëv [Mn] proved an even stronger theorem. When translated into complexity theory terms, his theorem implies that determining the stretchability of pseudoline arrangements is equivalent to the existential theory of the reals. What he actually shows is the following stronger theorem: given any primary semialgebraic variety (i.e., the solution space of a set of equations and strict inequalities over the reals) there exists a pseudoline arrangement whose realization space has the same topology (more precisely, the semialgebraic variety and the realization space are stably equivalent). For more details, see **[Mn]**. If the semialgebraic variety is empty (i.e., the equations and inequalities have no solution) the pseudoline arrangement will not be stretchable. To show that this implies the complexity result, all that is necessary is to show how to find this pseudoline arrangement in polynomial time. This can be done by carefully following Mnëv's proof **[Po]**. We will show how to find such a pseudoline arrangement using a variant of his proof which is not as topological, and is thus easier for computer scientists to understand.

Mnëv uses an intermediate step to reduce stretchability of pseudoline arrangements to the existential theory of the reals. That is, he reduces the existential theory of the reals to an intermediate problem, which he then reduces to the stretchability of pseudoline arrangements. We will call the intermediate problem "the existential theory of totally ordered real variables." The problem is:

> Given a set of variables $x_1, x_2, x_3, \ldots, x_n$, a set of equations on these variables of the forms
>
> $$x_i + x_j = x_k, \qquad x_i \times x_j = x_k,$$
>
> and the inequalities
>
> $$1 = x_1 < x_2 < x_3 < x_4 < \cdots < x_n,$$
>
> does there exist a set of real numbers satisfying these equations and inequalities?

Instead of proving directly that this problem is equivalent to the stretchability of pseudoline arrangements, we will show that it is equivalent to the realizability of point configurations, as Mnëv does. A *point configuration* is a set of points together with an orientation on every triple of points (i.e., we know whether each ordered triple is clockwise, counterclockwise, or collinear). Because line arrangements and point configurations are related by projective duality, stretchability of line arrangements is equivalent to realizability of point configurations. We produce a point configuration which is realizable if and only if there is a vector of real numbers satisfying our equations and inequalities.

We first show how to reduce the existential theory of totally ordered real variables to realizability of point configurations and then we show how to reduce the existential theory of the reals to the existential theory of totally ordered real variables.

For the first reduction (to realizability of point configurations) we use Mnëv's proof. We first place three lines in the plane. We can assume without loss of generality that these are the x-axis, the y-axis, and the line at infinity. (Otherwise we apply a projective transformation to obtain these.)

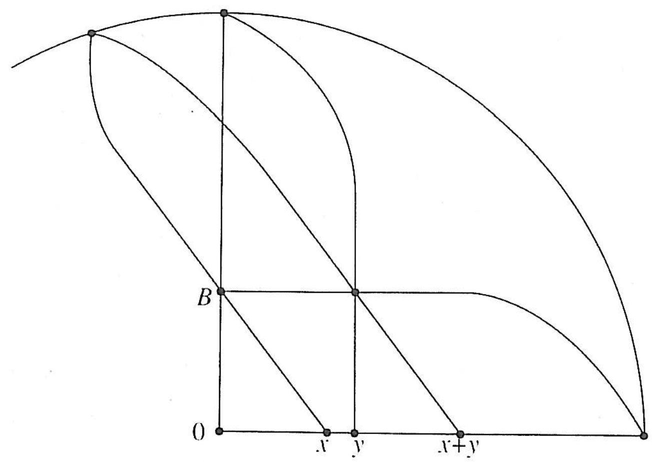

FIGURE 15. Addition in Mnëv's construction. The large circle is the line at infinity.

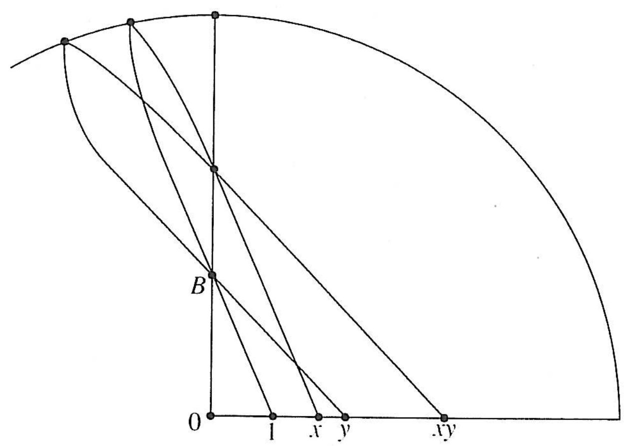

FIGURE 16. Multiplication in Mnëv's construction. The large circle is the line at infinity.

We next place points $P_{x_1}, P_{x_2}, \dots, P_{x_n}$, corresponding to our variables, on the x-axis. The x-coordinate of point P_{x_i} will be the value of the corresponding variable x_i. To perform an addition we introduce the set of points in Figure 15, and to perform a multiplication we introduce the set of points in Figure 16. The y-coordinate of point B in Figures 15 and 16 will be different for each equation; we will denote these points by $B_1, B_2, \dots,$ in the order that we place these equations in the point configuration. Performing additions and multiplications in a similar manner is an old technique; Mnëv's contribution was to realize that if the multiplications and additions are done in this manner, and point B_i in Figures 15 and 16 is placed sufficiently closer to y_∞ than points $B_1, B_2, \dots, B_{i-1}$, then the resulting line arrangement has a unique combinatorial structure, and thus the realizability of the point configuration is equivalent to the solvability of these equations and inequalities.

The next step is to reduce the existential theory of the reals to the existential theory of totally ordered real variables. We do this in three steps. The first is by reducing the existential theory of the reals to the problem: Given a set of equations and inequalities in the forms

$$x_i + x_j = x_k, \qquad x_i \times x_j = x_k, \qquad x_i < x_j,$$

is there a solution?

One can take any polynomial equation and reduce it to an equation of this form just by introducing variables for all the intermediate steps and building the polynomial term by term. For example, the equation $x^5 + y^2 = 2$ reduces to the following set of equations (where new variables are introduced by V's, with subscripts standing for the expression which the V's are supposed to represent):

$$V_{x^5} + V_{y^2} = 2, \qquad V_{x^5} = V_{x^4} \times x,$$
$$V_{x^4} = V_{x^2} \times V_{x^2}, \qquad V_{x^2} = x \times x, \qquad V_{y^2} = y \times y.$$

Constants can be built by starting with 1 and adding and multiplying to obtain integers, dividing to obtain rationals, and solving polynomial equations to obtain algebraic numbers. The number of such equations is clearly no more than a constant factor times the size of the input. Thus, we only require the basic constant 1. For complexity purposes, an inequality of the form $x \geq y$ can be taken care of by replacing it with the equation $x = y + V^2$, where V is a new variable. For the topological equivalence to hold, it seems that we need to restrict ourselves to strict inequalities (i.e., *primary* semialgebraic varieties).

We now reduce this problem to the same problem of determining if a set of equations of the above form has a solution, but with the additional restriction that all the variables be greater than 1. To do this, we replace each variable x_i with a variable V_{x_i+a}, which will be assumed to have the value x_i+a for some a, where a can be arbitrarily large. We also must introduce the variables V_a, V_{a^2}, and V_{a+a^2}, and the relations $V_a \times V_a = V_{a^2}$ and $V_a + V_{a^2} = V_{a+a^2}$. We now show that by introducing a few extra equations, we can add and multiply using only these new variables.

Comparisons are easy, since $x_i < x_j$ is equivalent to $V_{x_i+a} < V_{x_j+a}$. To add, we replace $x_i + x_j = x_k$ with

$$V_{x_i+a} + V_{x_j+a} = V_{x_i+x_j+2a}, \qquad V_{x_i+x_j+2a} = V_{x_k+a} + V_a.$$

It is easy to see that if we let $x_i = V_{x_i+a} - V_a$, and similarly obtain x_j, x_k, then the above equations imply that $x_i + x_j = x_k$.

Multiplying is somewhat more complicated. Instead of $x_i \times x_j = x_k$ we use

$$\begin{aligned} V_{x_i+a} \times V_{x_j+a} &= V_{x_ix_j+ax_i+ax_j+a^2}, \\ V_a \times V_{x_i+a} &= V_{ax_i+a^2}, \\ V_a \times V_{x_j+a} &= V_{ax_j+a^2}, \\ V_{ax_i+a^2} + V_{ax_j+a^2} &= V_{ax_i+ax_j+2a^2}, \\ V_{x_ix_j+ax_i+ax_j+a^2} + V_{a+a^2} &= V_{x_ix_j+ax_i+ax_j+a+2a^2}, \\ V_{ax_i+ax_j+2a^2} + V_{x_k+a} &= V_{x_ix_j+ax_i+ax_j+a+2a^2}. \end{aligned}$$

It is again easy to see that these equations force the desired relation $x_k = x_ix_j$ to hold.

Thus, by replacing variables x_j with variables V_{x_j+a}, and replacing equations and inequalities as described above, we obtain an equivalent set of equations and inequalities. It is easy to see that they are equivalent, because any solution for the x_i can be turned into a solution for V's satisfying $V > 1$ for all V's by simply choosing $a > |x_i| + 1$ for all x_i. Similarly, one can obtain the x_i from a solution for the V's by letting $x_i = V_{x_i+a} - V_a$.

Finally, we show how to obtain an ordered set of variables. The idea is essentially the same as the previous one, but somewhat more complicated. Again, we introduce a new variable, this time b. We make b larger than any of the previous variables (these were previously denoted by V_j but will now be denoted by $x_1, x_2, \ldots, x_n$). Now, we will work with a variable $V_{x_i+b^i} = x_i + b^i$ instead of with the original x_i's. Note that we are now using a different power of b for each variable. To obtain the powers of b, we introduce the equations $V_{b^{i+1}} = V_{b^i} \times V_b$ for $1 \le i \le r$, where we will determine the value of r later. Since $b > x_i$ for all x_i, we know the ordering of the variables $V_{x_i+b^i}$.

To add, multiply, and compare the $V_{x_i+b^i}$'s, we again choose different powers of b, at most three for each equation or inequality. The idea is to choose several unused powers of b, say b^α, for each equation, and work with $V_{x_i+b^\alpha}$ instead of $V_{x_i+b^i}$. The easiest case is again the inequality $x_i < x_j$. For this, we choose an unused power of b, say $\alpha > n$, and introduce the equations and inequalities

$$\begin{gathered} V_{b^\alpha} = V_{b^i} + V_{b^\alpha - b^i}, \qquad V_{b^\alpha} = V_{b^j} + V_{b^\alpha - b^j}, \\ V_{x_i+b^i} + V_{b^\alpha - b^i} = V_{x_i+b^\alpha}, \qquad V_{x_j+b^j} + V_{b^\alpha - b^j} = V_{x_j+b^\alpha}, \\ V_{x_i+b^\alpha} < V_{x_j+b^\alpha}. \end{gathered}$$

The last inequality will not be in our final set of equations because it is implicit in the ordering of the variables.

We need to check that we know the ordering of all the new variables we just introduced. This is true: if $i < j$, then

$$V_{b^\alpha - b^j} < V_{b^\alpha - b^i} < V_{b^\alpha} < V_{x_i + b^\alpha} < V_{x_j + b^\alpha} .$$

If $j < i$, the order of $V_{b^\alpha - b^i}$ and $V_{b^\alpha - b^j}$ is reversed. Because these variables are the only ones with a V_{b^α} term in them, we know their relative ordering with respect to the other variables in our problem.

To perform addition, suppose we have the equation $x_i + x_j = x_k$. We first choose three unused powers of b, say b^α, b^β, and b^γ, with $n < \alpha < \beta < \gamma$. We then introduce the equations

$$\begin{aligned} V_{b^\alpha} &= V_{b^i} + V_{b^\alpha - b^i} , \\ V_{b^\beta} &= V_{b^j} + V_{b^\beta - b^j} , \\ V_{b^\gamma} &= V_{b^k} + V_{b^\gamma - b^k} , \end{aligned}$$

$$\begin{aligned} V_{x_i + b^i} + V_{b^\alpha - b^i} &= V_{x_i + b^\alpha} , \\ V_{x_j + b^j} + V_{b^\beta - b^j} &= V_{x_j + b^\beta} , \\ V_{x_k + b^k} + V_{b^\gamma - b^k} &= V_{x_k + b^\gamma} ; \end{aligned}$$

$$\begin{aligned} V_{b^\gamma} &= V_{b^\alpha} + V_{b^\gamma - b^\alpha} , \\ V_{b^\gamma - \alpha} &= V_{b^\beta} + V_{b^\gamma - b^\alpha - b^\beta} , \\ V_{x_i + b^\alpha} + V_{x_j + b^\beta} &= V_{x_i + x_j + b^\alpha + b^\beta} , \\ V_{x_i + x_j + b^\alpha + b^\beta} + V_{b^\gamma - b^\alpha - b^\beta} &= V_{x_k + b^\gamma} . \end{aligned}$$

The first six equations produce variables of the form $V_{x_i + b^\alpha}$ instead of the form $V_{x_i + b^i}$. The remaining four do the work of adding x_i and x_j. We know the ordering of the variables, because

$$\begin{aligned} V_{b^\alpha - b^i} < V_{b^\alpha} &< V_{x_i + b^\alpha} , \\ V_{b^\beta - b^j} < V_{b^\beta} < V_{x_j + b^\beta} &< V_{x_i + x_j + b^\beta + b^\alpha} , \\ V_{b^\gamma - b^\alpha - b^\beta} < V_{b^\gamma - b^\alpha} < V_{b^\gamma - b^k} < V_{b^\gamma} &< V_{x_k + b^\gamma} . \end{aligned}$$

Multiplication is, as usual, the most complicated operation. As before, we will choose three unused powers of b, b^α, b^β, and b^γ, and $n < \alpha < \beta < \gamma$, but this time we also require that $\gamma = \alpha + \beta$. We now introduce the same first six equations as for the case of addition, as well as the equations

$$V_{b^\gamma} + V_{b^\gamma} = V_{2b^\gamma},$$
$$V_{x_i+b^\alpha} \times V_{x_j+b^\beta} = V_{x_ix_j+b^\beta x_i+b^\alpha x_j+b^\gamma},$$
$$V_{x_ix_j+b^\beta x_i+b^\alpha x_j+b^\gamma} + V_{2b^\gamma} = V_{x_ix_j+b^\beta x_i+b^\alpha x_j+3b^\gamma},$$
$$V_{x_i+b^\alpha} \times V_{b^\beta} = V_{b^\beta x_i+b^\gamma},$$
$$V_{x_j+b^\beta} \times V_{b^\alpha} = V_{b^\alpha x_j+b^\gamma},$$
$$V_{b^\alpha x_j+b^\gamma} + V_{b^\beta x_i+b^\gamma} = V_{b^\alpha x_j+b^\beta x_i+2b^\gamma},$$
$$V_{x_ix_j+b^\beta x_i+b^\alpha x_j+3b^\gamma} = V_{b^\alpha x_j+b^\beta x_i+2b^\gamma} + V_{x_k+b^\gamma}.$$

Again, it is easily checked that we know the ordering of the variables.

By going through the above proof with more care, we can show that if the semialgebraic variety was defined only by inequalities, then we can produce an equivalent *uniform* point configuration. We do this by showing that the point configuration we obtain is constructible, and then applying Lemma 4 (this time in the dual version for points and not lines, as it is used in [**Mn, SW**]).

The above proof also gives the original version of Mnëv's theorem. We have to show that the solution space of our set of equations on ordered variables is topologically equivalent to the solution space of the semialgebraic variety. This follows in the first (or second) reduction if, given a solution with some value of a (or b), we can always increase a (or b) and still have a solution. This can easily seen to be the case for a. For b, this is not clear; however, we can show this holds for b if we introduce some extra variables and equations. We introduce the extra variables V_{b^i+1} and V_{b^i+b} for $1 < i < n$, along with the corresponding equations that force them to have the appropriate values, thus ensuring that $V_{b^i+1} < V_{x_i+b^i} < V_{b^i+b}$, so $1 < x_i < b$. We also introduce another equation ensuring that $b > 6$. Now, if we choose any $b > \max(6, x_1, x_2, \dots, x_n)$ we can show that for any set of x_i, given a solution, this b leads to a set of $V_{f(x_i, b)}$ which is a solution of the equations with ordered variables.

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References

[**BS**] J. Bokowski and B. Sturmfels, *Computational synthetic geometry*, Lecture Notes in Math., vol. 1355, Springer, Berlin, 1989.

[**GJ**] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, Freeman, New York, 1979.

[**Gr**] B. Grünbaum, *Arrangements and spreads*, Regional Conf. Ser. in Math., Amer. Math. Soc., Providence, RI, 1972.

[Mn] N. E. Mnëv, *The universality theorems on the classification problem of configuration varieties and convex polytopes varieties*, in Topology and Geometry—Rohlin Seminar (O. Y. Viro, ed.), Lecture Notes in Math., vol. 1346, Springer, Berlin, 1988, pp. 527–544.

[Po] R. Pollack, Private communication.

[SW] B. Sturmfels and N. White, *Constructing uniform oriented matroids without the isotopy property*, IMA Preprint Ser. **32** (1988).

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