Uniqueness of the maximal entropy measure on essential spanning forests

A One-Act Proof by Scott Sheffield

First, we introduce some notation...

An **essential spanning forest** of an infinite graph G is a spanning subgraph F of G, each of whose components is a tree with infinitely many vertices. For simplicity, we fix $G = \mathbb{Z}^d$. The number of **topological ends** of an infinite tree T is the maximum number of disjoint semi-infinite paths in T.

Ω: the set of essential spanning forests of G.

F_H: the set of edges of F contained in a subgraph H of G.

$\mathcal{F}$: the standard product σ-algebra (i.e., the smallest σ-algebra in which the functions $F \to F_H$ are measurable).

An shift-invariant measure on $(\Omega, \mathcal{F})$ is **ergodic** if it is an extreme point of the set of shift-invariant measures on $(\Omega, \mathcal{F})$.

and some more notation...

μ_G: the weak limit of the uniform measures on spanning trees of G_n, where G_n are any increasing sequence of subgraphs of G whose union is G. This limit exists independently of the choice of G_n and is translation invariant and ergodic (Pemantle and Benjamini, Lyons, Peres, Schramm).

Write $\Lambda_n = [-n, n]^d$. The **specific entropy** (a.k.a. **entropy per site**) of μ, which we denote $\mathrm{Ent}(\mu)$, is

$$-\lim_{n\to\infty} |V(\Lambda_n)|^{-1} \sum \mu(\{F_{\Lambda_n} = F_n\})$$

$$\log \mu(\{F_{\Lambda_n} = F_n\})$$

where the sum ranges over all spanning subgraphs F_n of Λ_n for which $\mu(\{F_{\Lambda_n} = F_n\}) \neq 0$. This limit always exists. (Can be proved with subadditivity, or by citing more general results, e.g., Ornstein and Weiss.)

and state the main result...

THEOREM: The measure μ_G is the unique ergodic probability measure on $(\Omega, \mathcal{F})$ with maximal specific free entropy. (In fact, theorem holds for any amenable and quasi-transitive G.)

Burton and Pemantle (1993): proved this when $G = \mathbb{Z}^d$ (and slightly more generally). Highlighted case $d = 2$, where result was equivalent to uniqueness of maximal entropy measure on domino tilings of the plane (perfect matchings of $\mathbb{Z}^2$).

Lyons (2002): discovered error in proof of Burton and Pemantle; asked about general amenable, quasi-transitive graphs (our result).

and recall some background results.

BACKGROUND FACT 1 (BP, L, P): The measure μ_G **has maximal specific entropy**. Moreover, Ent(μ_G) is the limit of the entropy-per-site of the uniform measure on spanning trees of Λ_n.

BACKGROUND FACT 2 (BLPS, BP, P): Partially-wired-boundary USTs converge to μ_G: Let $G_n \to G$ and for each n, consider any equivalence relation on the boundary vertices of G_n, and let G'_n be the graph obtained from G_n by identifying equivalent vertices. Then the uniform measures on spanning trees of G'_n converge weakly to μ_G. In particular, this holds for both the **wired boundary** (all boundary vertices identified) and **free boundary** (no boundary vertices identified).

BACKGROUND FACT 3 (BLPS, BK, P): If μ is shift-invariant, then μ-almost surely all trees in F have **at most two topological ends**.

Burton-Pemantle Strategy: show that each maximal entropy μ has a strong Gibbs property and...

STRONG GIBBS PROPERTY: Fix any finite induced subgraph H of G, and write $a \sim_O b$ if there is a path from a to b consisting of edges *outside* of H. Let H' be the graph obtained from H by identifying vertices equivalent under $\sim_O$. Then given $F_{G \setminus H}$, the conditional measure on F_H is the UST measure on H'.

that if μ has the strong Gibbs property, then $\mu = \mu_G$.

Unfortunately...

The maximal entropy μ may not have the strong Gibbs property. For example, if $G = \mathbb{Z}^d$ with $d > 4$, then $\mu_G \in \mathcal{E}_G$ and μ_G almost surely F contains infinitely many trees, each of which has only one topological end (P, BLPS). Thus, conditioned on $F_{G \setminus H}$, all configurations F_H that contain paths joining distinct infinite trees of $F_{G \setminus H}$ have probability zero.

Our strategy: show that each maximal entropy μ has a weaker Gibbs property and...

WEAK GIBBS PROPERTY: For each a and b on the boundary of H, write $a \sim_I b$ if a and b are connected by a path contained *inside* H. Then conditioned on this relation and $F_{G \setminus H}$ all choices for F_H which give the same relation occur with equal probability.

if μ has the weak Gibbs property then and there is μ a.s. at most one two-ended tree, then $\mu = \mu_G$.

CASE 1: If μ has the weak Gibbs property and μ almost surely all trees have only one topological end, in which case $\mu = \mu_G$,

CASE 2: If μ has the weak Gibbs property and μ-almost surely F consists of a single two-ended tree, then $\mu = \mu_G$.

CASE 3: If μ has the weak Gibbs property, and μ-almost surely F contains exactly one two-ended tree, then μ almost surely F consists of a single tree and $\mu = \mu_G$.

If μ has weak Gibbs property and with positive probability there are multiple two-ended trees, then $\mathrm{Ent}(\mu) < \mathrm{Ent}(\mu_G)$.