DOMINO TILINGS

and their

INVARIANT GIBBS MEASURES

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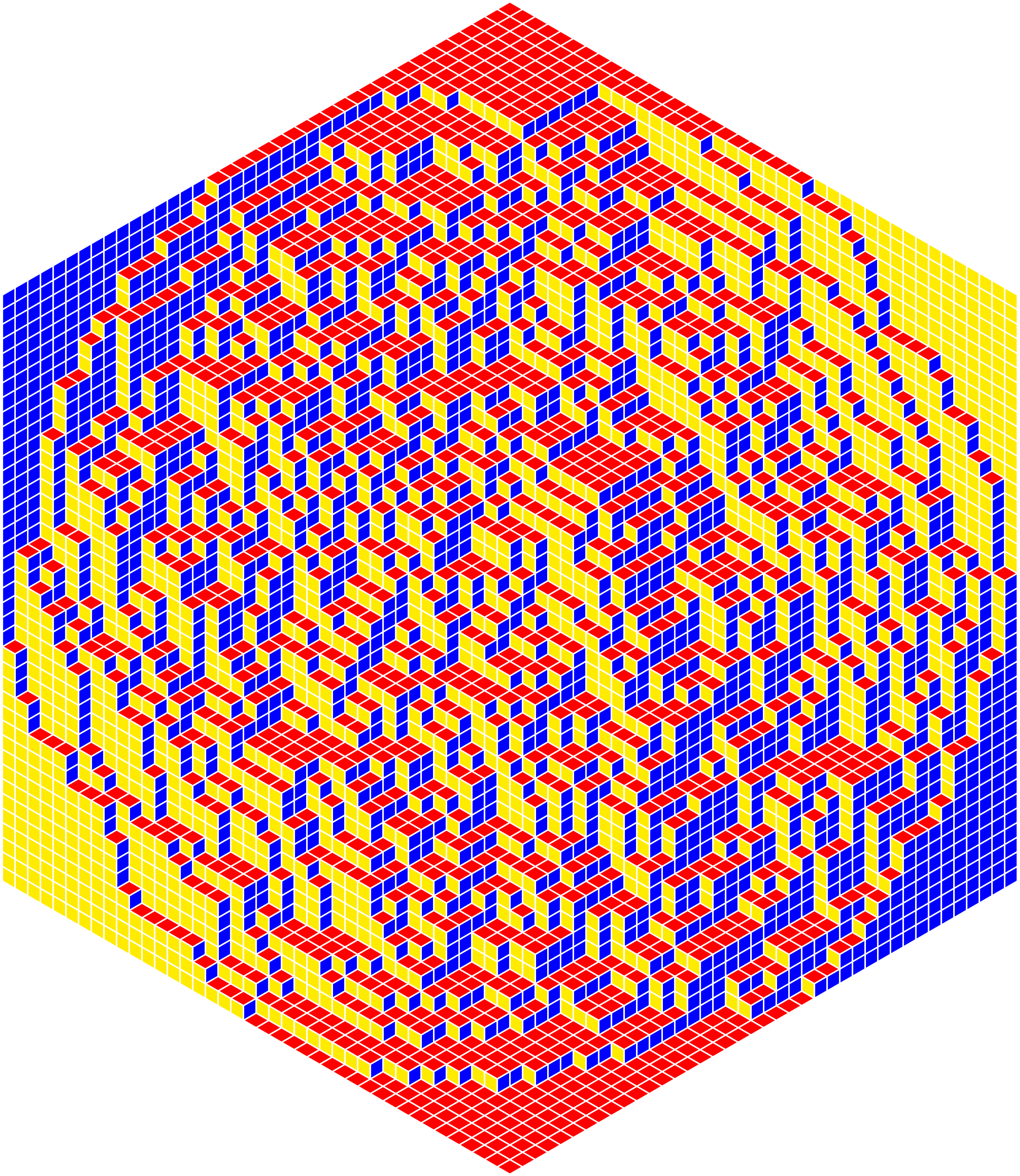
References on arXiv.org

1. *Random Surfaces*, to appear in *Asterisque*.
2. *Dimers and amoebae*, joint with Kenyon and Okounkov, to appear in *Annals of Mathematics*.
3. *Maximal entropy measures on essential spanning forests*, to appear in *Annals of Probability*.



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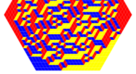

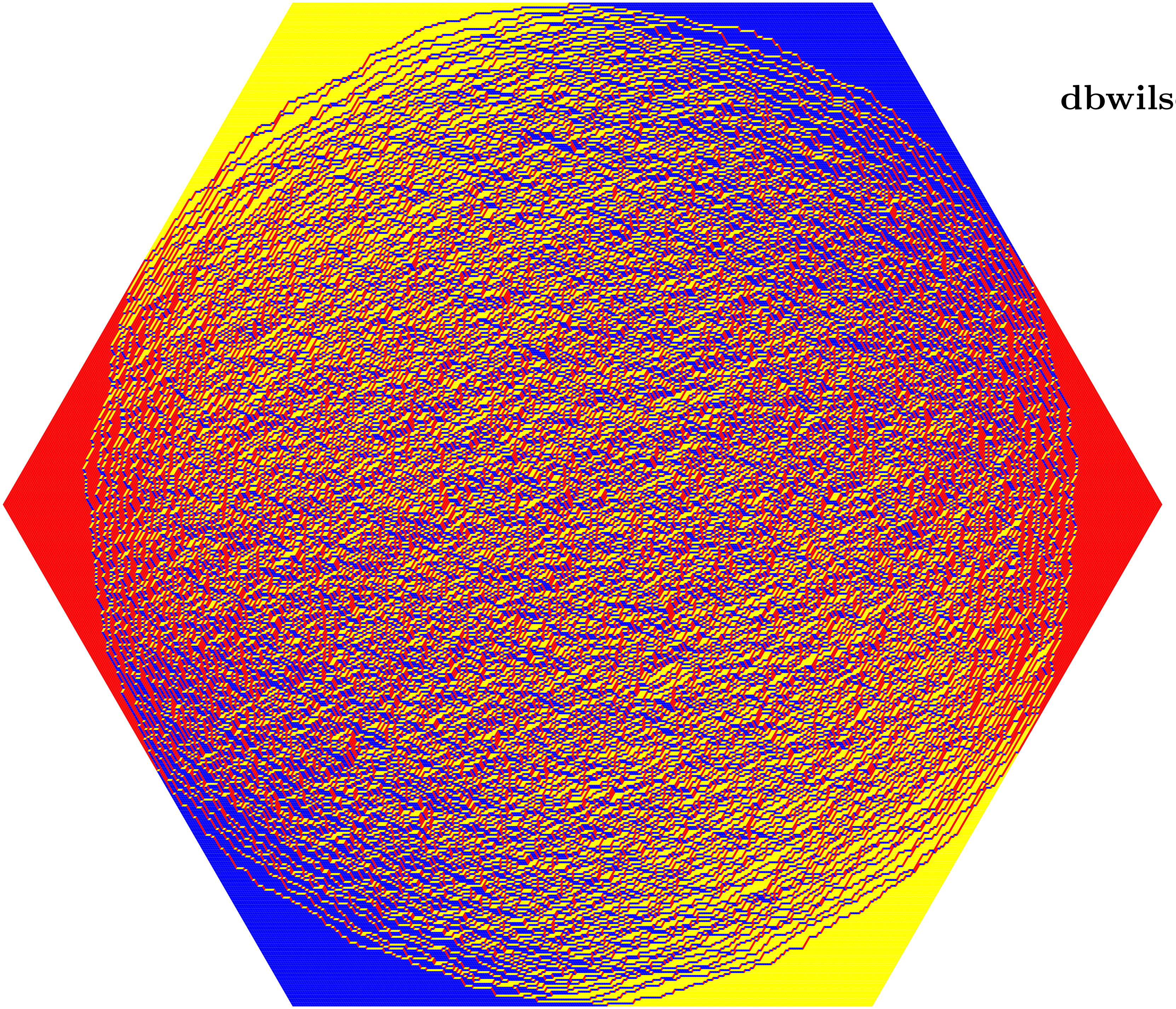

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Height functions

Let Ω be the set of all valid height functions $\phi : \mathbb{Z}^2 \to \mathbb{R}$, i.e., functions with differences $3/4$ or $-1/4$ when crossing any edge (black vertex on left) and $\phi((0,0)) \in \mathbb{Z}$. Let $\mathcal{F}$ be the product σ-algebra on Ω and $\mathcal{F}^\tau$ the smallest sub-σ-algebra in which "differences in height" of the form $\phi(y) - \phi(x)$ are measurable.

A measure μ on $(\Omega, \mathcal{F})$ is called a measure on heights.

A measure μ on $(\Omega, \mathcal{F}^\tau)$ is called a measure on tilings.

Gibbs measures

A measure μ on $(\Omega, \mathcal{F}^\tau)$ is called a gradient Gibbs measure or Gibbs measure on tilings if conditioned on the tiles outside of a region R, all ways of extending the tiling to that region are equally likely.

A Gibbs measure μ is extremal if it is an extreme point of the convex set of all Gibbs measures or, equivalently, if it satisfies a zero one law on tail events.

Let $\mathcal{L}$ be the set of chessboard-coloring preserving translations of $\mathbb{Z}^2$. An $\mathcal{L}$-invariant measure μ is $\mathcal{L}$-ergodic if it is an extreme point of the convex set of $\mathcal{L}$-invariant measures or, equivalently, if μ satisfies a zero-one law on $\mathcal{L}$-invariant events.

Important facts

1. Every Gibbs measure can be uniquely written as a weighted average of *extremal* Gibbs measures.
2. Every $\mathcal{L}$-invariant measure can be uniquely decomposed into $\mathcal{L}$-ergodic measures. A measure μ is Gibbs if and only if its $\mathcal{L}$-ergodic components are a.s. Gibbs.
3. If μ is extremal, and Λ_n are increasing finite subsets of $\mathbb{Z}^2$ whose union is $\mathbb{Z}^2$, then $\mu = \lim \phi\gamma_{\Lambda_n}$ for almost all ϕ. (Here γ_{Λ_n} is the usual "rerandomize on Λ_n" kernel.)

In other words...

To sample from an $\mathcal{L}$-invariant measure μ on tilings, you can

1. First sample an **$\mathcal{L}$-ergodic component** μ_1 of μ.
2. Then sample an **extremal component** μ_2 of μ_1.
3. Then sample an actual **tiling** from μ_2.

Another important fact

We say a Gibbs measure $(\Omega, \mathcal{F}^\tau)$ is smooth if it can be realized as the restriction to $\mathcal{F}^\tau$ of a Gibbs measure on $(\Omega, \mathcal{F})$. Clearly, a Gibbs measure is smooth if and only if almost all of its extremal components are smooth. If μ is an extremal Gibbs measure $(\Omega, \mathcal{F})$, then the μ probability distribution on height at a point is log concave — in particular, it has moments of all order.

Slope

Define slope of an $\mathcal{L}$-invariant μ by

$$S(\mu) = (\mu\,(\phi(2,0) - \phi(0,0))\,, \mu\,(\phi(0,2) - \phi(0,0)))\,.$$

Note that $S(\mu) = (P_S - P_N, P_E - P_W)$ where P_N, P_S, P_E, P_W are densities of dominos that point north, south, east west from their black square. The four "brickwork" singletons have slopes $(\pm 1, 0)$ and $(0, \pm 1)$. All possible slopes lie in convex hull of these; let U be its interior.

Q: What are the $\mathcal{L}$-ergodic Gibbs measures?

1. 1965: Kasteleyn publishes *The statistics of dimer arrangements on a lattice*.
2. 1975: Messager and Miracle-Solé classify ergodic Gibbs measures for the Ising model.
3. 1993: Burton and Pemantle prove uniqueness of zero slope ergodic Gibbs measure using spanning tree result (error found by Lyons, 2002; new proof by S, 2004).
4. 1996: Cohn, Elkies, and Propp conjecture existence of unique $\mathcal{L}$-ergodic, slope u Gibbs measure for $u \in U$.
5. 2000: Cohn, Kenyon, Propp give explicit formulae for slope u Gibbs measures in domino tiling case.
6. 2003: Kenyon, Okounkov, S give explicit formulae for general weighted doubly periodic lattices involving amoebae of Harnack curves.

A: Exists a unique one of each slope $u \in U$.

1. We will give a complete classification of the ergodic Gibbs measures of a given slope u. The hard part will proving the Cohn, Elkies, Propp conjecture—that for each $u \in U$ there is a unique ergodic Gibbs measure of slope u.

2. An adaptation of the Edwards-Sokal, Fortuin-Kasteleyn, Swendsen-Wang updates called cluster swapping enables us to extend the proof to any continuum or discrete height model with convex nearest neighbor potential (e.g., square ice, linear solid-on-solid models).

3. The approach also yields a soft proof for general models that "crystal facets" have slopes in the dual of the lattice of translation invariance. Algebraic proof given by Kenyon, Okounkov, S for periodic dimer models.

Easy part: When $S(\mu) \in \partial U$

When the slope lies on the boundary, there are at most two directions (e.g., south and east) that the dominos can point. Every SW to NE diagonal row of black squares has all of its dominos point south or all of them point east. The $\mathcal{L}$-ergodic measures are thus in one-to-one correspondence with one dimensional ergodic measures on $\{S, E\}^{\mathbb{Z}}$.

Next: Variational Principle

Write μ_Λ for restriction of μ to the finite set $T_1, \ldots T_n$ of tilings of Λ. Write

$$\mathrm{FE}_\Lambda(\mu) = \sum_{i=1}^{n} \mu(T_i) \log \mu(T_i)$$

$$\mathrm{SFE}(\mu) = \lim_{n\to\infty} |\Lambda_n|^{-1} FE_{\Lambda_n}(\mu),$$

called the specific free energy or μ.

Variational Principle

If μ is an $\mathcal{L}$-ergodic measure on tilings, then μ is a Gibbs measure if and only if $\mathrm{SFE}(\mu)$ is minimal among all $\mathcal{L}$-invariant measures with slope $S(\mu)$. Make an analogous definition if μ is a measure on pairs of tilings that is invariant under translations that translate both components in tandem.

If μ has marginals μ_1, μ_2 then $\mathrm{SFE}(\mu) \geq \mathrm{SFE}(\mu_1) + \mathrm{SFE}(\mu_2)$. If μ is an $\mathcal{L}$-ergodic measure on pairs of tilings and the average slope of the components of μ is u and

$$\mathrm{SFE}(\mu) = SFE(\mu_u \otimes \mu_u) = 2\mathrm{SFE}(\mu_u),$$

where μ_u is an $\mathcal{L}$-ergodic Gibbs measure of slope u, then μ is Gibbs.

Infinite path lemma

Lemma: If μ_1 and μ_2 are extremal Gibbs measures on tilings and $\mu_1 \otimes \mu_2$ a.s. the union of the two tilings contains no infinite path, then $\mu_1 = \mu_2$.

Couplings

Suppose that μ_1 and μ_2 are distinct ergodic Gibbs measures, both of slope $u \in U$. How many infinite paths are in the union of tilings sampled from $\mu_1 \otimes \mu_2$? Previous lemma rules out zero. We now rule out having k infinite paths with positive probability when

1. when $2 \leq k < \infty$ (swapping)
2. when $k = \infty$ (Burton-Keane and swapping)
3. when $k = 1$ (height offsets, RSW, homotopy of countably punctured plane)

Swapping infinite paths

Let $R(\mu_1 \otimes \mu_2)$ be measure on tilings pairs defined as follows: to sample, first sample (ϕ_1, ϕ_2) from $\mu_1 \otimes \mu_2$. Then pick one of the infinite paths in the union and flip a coin to decide its orientation (whether height goes up or down by one when crossing it). For every other infinite path, determine its orientation by requirement that $\phi_2 = \phi_1$ assume only two integer values (say, zero and one) on infinite components.

If $2 \leq k \leq \infty$...

FACT: unless $u \in \partial U$ it is a.s. possible to join together all of the infinite zero clusters (or all of the one clusters) with finitely many local changes to the two tilings. Hence the number of infinite clusters with height zero is an $\mathcal{L}$-invariant function on tiling pairs such that μ, conditioned on this function assuming some value, is not a Gibbs measure.
Hence $R(\mu_1 \otimes \mu_2)$ is not Gibbs. However, it is easily seen that

$$\mathrm{SFE}(R(\mu_1 \otimes \mu_2)) = \mathrm{SFE}(\mu_1 \otimes \mu_2).$$

Hence this has minimal specific free energy given its slope—since the swappings don't change average slope and $\mathrm{SFE}(\mu_1 \otimes \mu_2)$ is minimal (by variational principle)—so variational principle implies $R(\mu_1 \otimes \mu_2)$ is Gibbs, a contradiction.

If $2 \leq k \leq \infty$...

For any three zero clusters, we can almost surely join the three together with finitely many local moves. Thus, for some Λ_n, there is a positive probability that there exists a connected component C_0—of the intersection of an infinite cluster C and Λ_n—whose removal breaks C into three infinite pieces.

The expected number of such "trifurcation clumps" in Λ_n grows linearly in $|\Lambda_n|$. However, Burton and Keane proved that the total possible number of disjoint trifurcation clumps in Λ_n is bounded above by $|\partial\Lambda_n|$, a contradiction.

If $k = 1$...

In this case, $\mu_1 \otimes \mu_2$ almost surely there exists *exactly* one infinite path in the union of the two perfect matchings.

Given sample (ϕ_1, ϕ_2) from $\mu_1 \otimes \mu_2$, we can define the **average height difference** to be the the limiting density of set of points on the side of the infinite path on which the height function of ϕ_2 minus that of ϕ_1 is largest. This difference is tail trivial property—so it is also defined on extremal components.

In this case, the μ_i are clearly smooth. We can show that the extremal components of μ_1 and μ_2 are indexed by the limit of the average expectation of heights on $n \times n$ grids centered at the origin. Call this limiting average height the height offset of the extremal measure.

Law on height offsets

For $i \in \{1, 2\}$ the μ_i probability distribution on set of height offsets modulo one is ergodic under adding $\frac{1}{2}(u, y)$ for $y \in \mathcal{L}$. If μ_1 and μ_2 are distinct, these probability distributions on $[0, 1)$ are mutual singular. In particular, μ_1 and μ_2 cannot be distinct if either component of u is irrational.

If u has rational coordinates, then there are finitely many height offsets; each ergodic Gibbs measure decomposes into finitely many components. Let ν_1 and ν_2 be extremal components from μ_1 and μ_2 respectively. These are invariant under a full rank sublattice of $\mathbb{Z}^2$. Normalize the height so that ν_1 and ν_2 both have height offsets in $[0, 1)$. Let Γ be the set of points on the high side of the path. It satisfies the **FKG inequality**, i.e., increasing functions of Γ are non-negatively correlated.

FKG inequality result

THEOREM: There exists no measure ρ on infinite simply connected subsets Γ of the squares of the $\mathbb{Z}^2$ lattice such that

1. ρ is invariant under a full rank sublattice of translations of $\mathbb{Z}^2$.
2. The boundary between Γ and its complement is almost surely an infinite path.
3. Increasing functions of Γ are non-negatively correlated.

Idea: take grid of boxes, look at possible topology of path in complement