TUG OF WAR

and the

INFINITY LAPLACIAN

How to solve **degenerate elliptic PDEs** and the **optimal Lipschitz extension problem** by playing games.

Yuval Peres, Oded Schramm, Scott Sheffield, and David Wilson

Infinity Laplacian on a graph:

$$\Delta_\infty u(x) = \frac{1}{2}\left(\inf_{y\sim x} u(y) + \sup_{y\sim x} u(y)\right) - u(x)$$

Infinity Laplacian in $\mathbb{R}^n$:

$$\Delta_\infty u(x) = \frac{\sum u_{x_i} u_{x_i x_j} u_{x_j}}{|\nabla u|^2} =$$

"2nd derivative of u in the gradient direction"

Say u is **infinity harmonic** if $\Delta_\infty u = 0$. Infinity harmonic functions are limits of p-harmonic functions (i.e., minimizers of $\int |\nabla u(x)|^p dx$) as $p \to \infty$. The p-harmonic functions solve the **Euler Lagrange equation**

$$\mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0$$

which can be rewritten as:

$$|\nabla u|^{p-2} (\Delta u + (p-2)\Delta_\infty u) = 0$$

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Outline

1. *Absolutely minimizing Lipschitz extensions (AMLEs)*

2. *Degenerate elliptic PDEs*

3. *Random Turn Hex* **and other selection games**

4. *Tug of war* **on a graph**

5. **Infinity harmonic functions on graphs: existence and uniqueness**

6. *Comparison with cones* **and AMLEs on length spaces**

Optimal Lipschitz Extensions

Given a Lipschitz function u, defined on a subset Y of a metric space X with metric δ, what is the **"tightest"** Lipschitz extension of u to all of X?

If "tightest" means "minimizing Lipschitz norm"

$$L_u(X) := \sup_{x,y \in X} \frac{|u(x) - u(y)|}{\delta(x,y)},$$

then (noting that we must have $L_u(X) \geq L_u(Y)$) the **McShane-Whitney extensions** (1934) are largest and smallest extensions achieving this bound:

$$\inf_{y \in Y} (u(y) + L_u(Y)|x - y|)$$

$$\sup_{y \in Y} (u(y) - L_u(Y)|x - y|)$$

AMLEs

Say $u : X \to \mathbb{R}$ is an **absolutely minimizing Lipschitz (AML)** extension of its values on Y if for any open $Z \subset (X \backslash Y)$ of finite diameter, we have

$$L_u(Z) \leq L_u(\partial Z)$$

Theorem [Aronsson, Jensen]: When X is a bounded, closed subset of $\mathbb{R}^n$, $Y = \partial X$, and u is a Lipschitz function on Y, there exists a **unique** AML extension of u to X.

EXISTENCE PROOF: G. Aronsson (1967) proves existence when X is bounded subset of $\mathbb{R}^n$, also shows that solutions are **infinity harmonic**, i.e., **"viscosity solutions" to $\Delta_\infty u = 0$**.

EXISTENCE EXTENSION: Juutinen (2002) extends existence to case that X is a **separable length space**.

UNIQUENESS PROOF 1: Jensen (1993)

UNIQUENESS PROOF 2: Barles and Busca (2001)

UNIQUENESS PROOF 3: Crandall, Aronsson, and Juutinen (2004): generalizes X to uniformly convex norms on $\mathbb{R}^n$.

We prove existence *and* uniqueness for *all* length spaces using a game called *Tug of War*.

Caselles, Masnou, Morel, and Sbert. **Image Interpolation.** Seminaire de l'Ecole Polytechnique, Palaiseau, Paris, 1998.

Figure 8:

Above Original image where occlusions are in white.

Below-left Disocclusion performed by solving equation $D^2u(\frac{Du}{|Du|}, \frac{Du}{|Du|}) = 0$. Singularities cannot be restored but regular parts of image are well recovered.

Below-right Disocclusion performed by the level lines based algorithm solving equation $D^2u(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}) = 0$. The singularities are well restored.

Family of *degenerate elliptic PDEs*

$$\sum \sigma_{i,j}(x, \nabla u) u_{x_i x_j} + \alpha(x, \nabla u) = 0$$

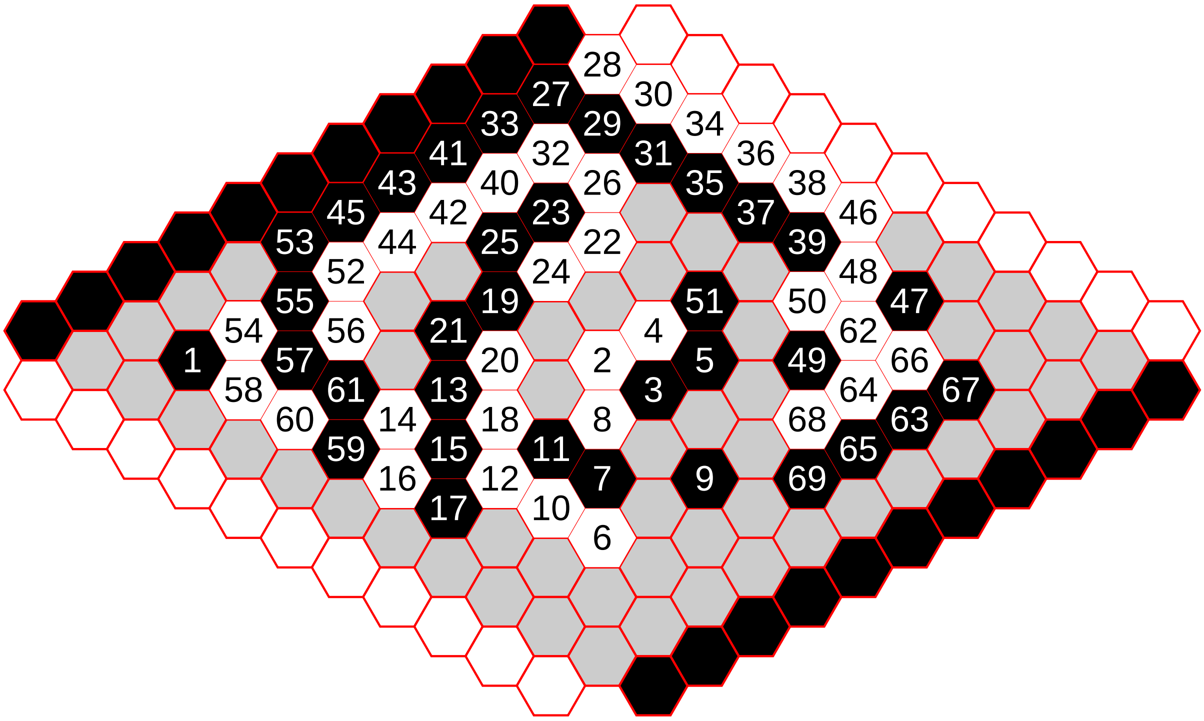
where σ and α depend continuously on x and ∇u and σ is always **positive semi-definite** but *not necessarily* positive definite).

Examples include Laplacian, p-Laplacian, and ∞-Laplacian.

Our results imply existence and uniqueness for solutions to a large class of PDEs in this family. In particular, for a fixed positive uniformly continuous g with Lipschitz boundary conditions, we prove existence and uniqueness of solutions to $\Delta_\infty u = g$. This is the first uniqueness result for $g \neq 0$. Uniqueness fails if g can be both positive and negative, and game theory tells us why.

Hex

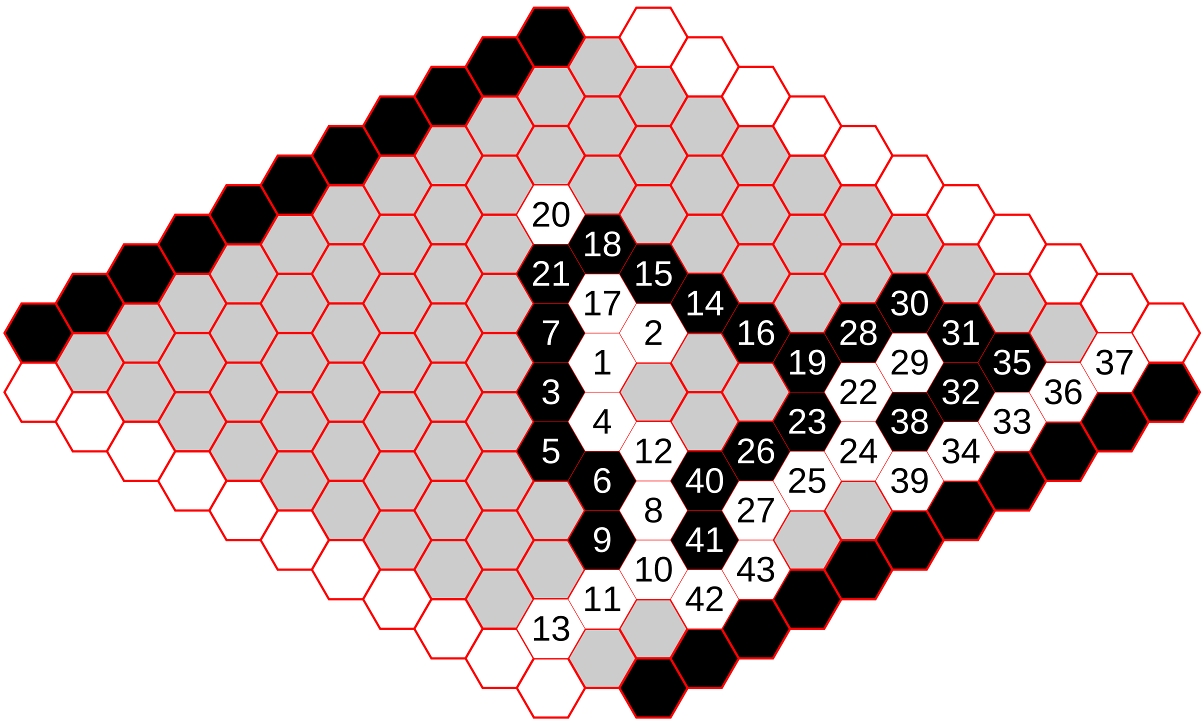
The game of hex was invented independently by Piet Hein in 1942 and John Nash in 1948.



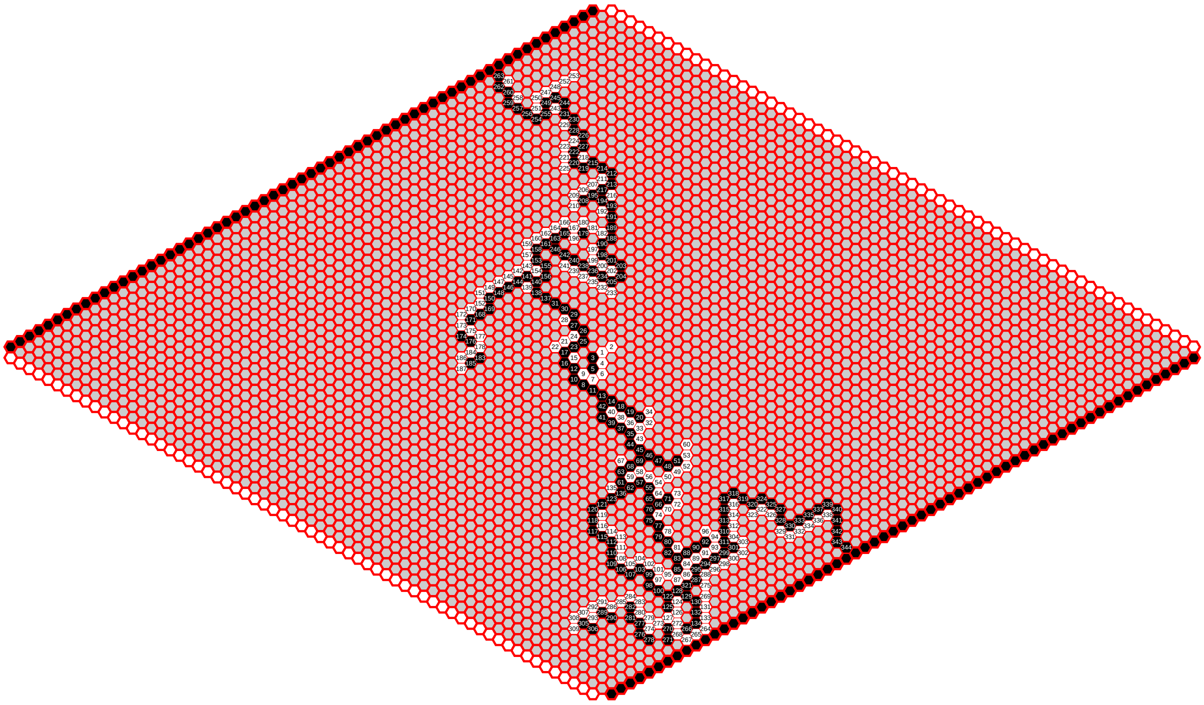
A game of hex played at the fifth computer olympiad in London, August 24, 2000.

Random turn hex

After a player moves, flip a fair coin to see who gets to move next. What is the optimal strategy?



Nearly optimal random turn hex



Cricket team selection

GIVEN: 2 captains, n players, a payoff function F from the set of 2^n player subsets to the probability the first captain's team wins.

CONVENTIONAL APPROACH: captains alternate choosing players until all players are chosen, and the teams play one game.

RANDOM TURNS: same except that before each choice, a fair coin toss determines which captain gets to choose a player.

CLAIM: If F is generic (say, its values are linearly independent over $\mathbb{Q}$) and both players play optimally, then each captain's final team will be uniformly distributed over the 2^n possibilities.

Tug of war

Tug of war on a graph: Given an graph G with marked subset T of terminal vertices and a payoff function $F : T \to \mathbb{R}$. A player who wins a coin toss may move to any neighbor of the current state. Game ends when terminal vertex is reached.

EXAMPLE: ϵ-step Planar Tug of War. The vertex set is $\mathbb{R}^2$ and $x \sim y$ if $|x - y| \leq \epsilon$, $T = T_1 \cup T_2$ (with F equal to 1 on T_1, 0 on T_2.)

Tug of war games (on undirected graphs) are a very general class of **reversible**, **player-symmetric** games.

Value existence

When the game starts at v, player one's **value**, denoted $V_1(v)$, is the

supremum, over all player one strategies, of the

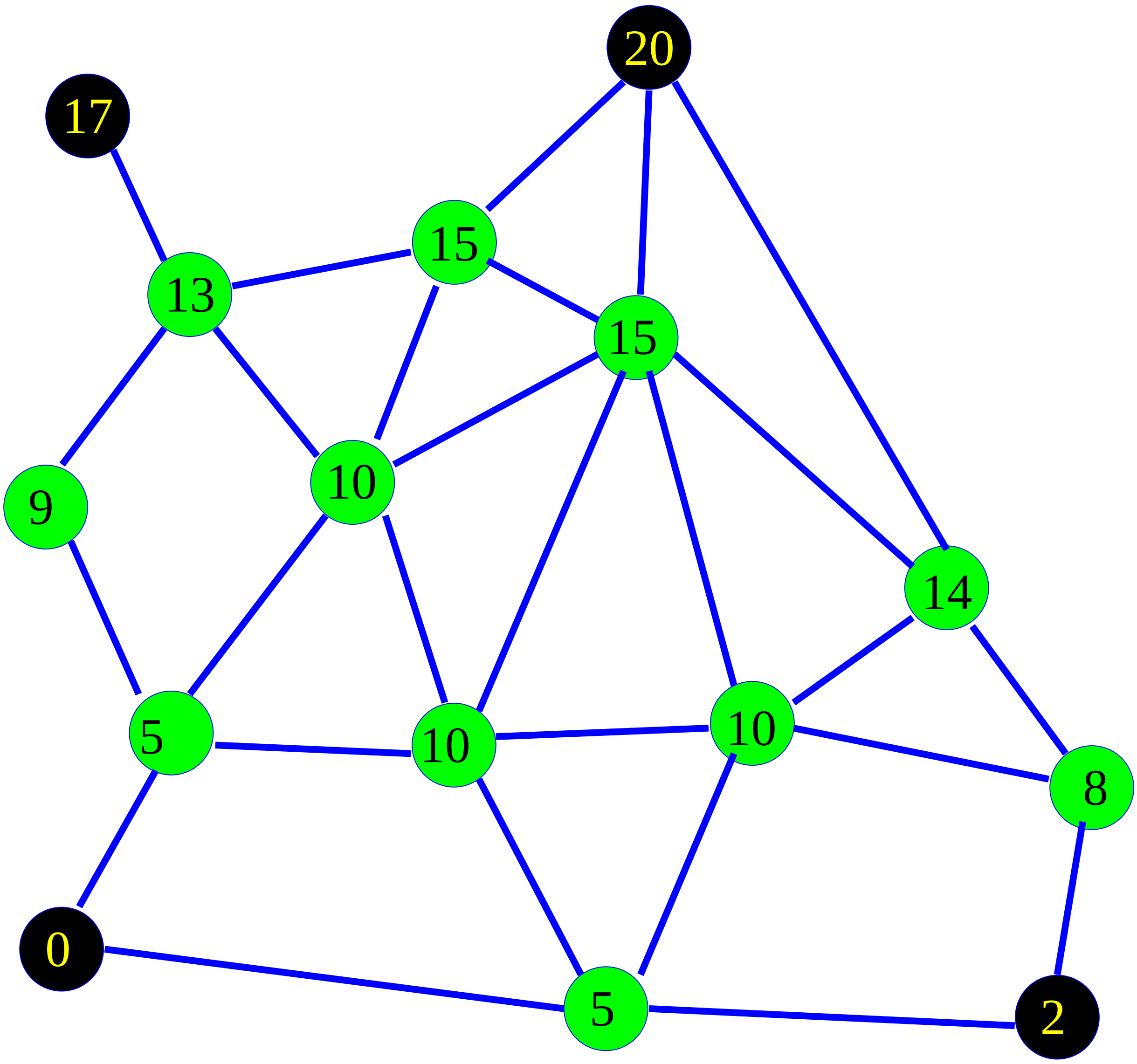
infimum, over all player two strategies, of the

expected payoff for player one when the players use those strategies (which we set equal to $-\infty$ if game does not end almost surely).

Define $V_2(v)$ similarly. Say game has a **value function** V if $V_1 = V_2$.

The functions V_1, V_2, and V are all **infinity harmonic**:

$$V(x) = \frac{1}{2}\left(\sup_{y\sim x} V(y) + \inf_{y\sim x} V(y)\right)$$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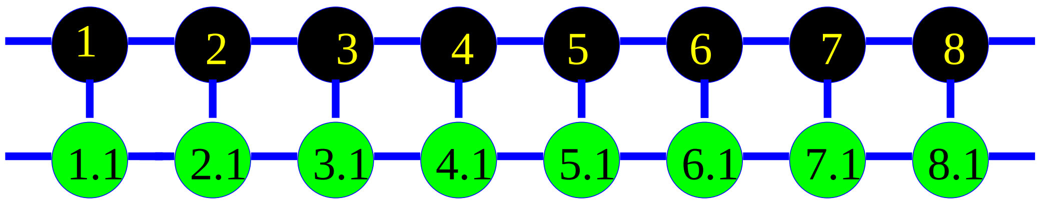
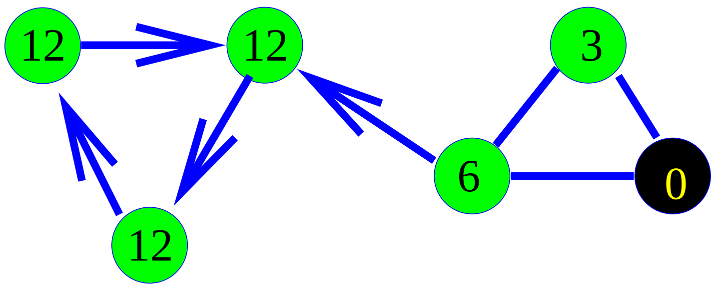
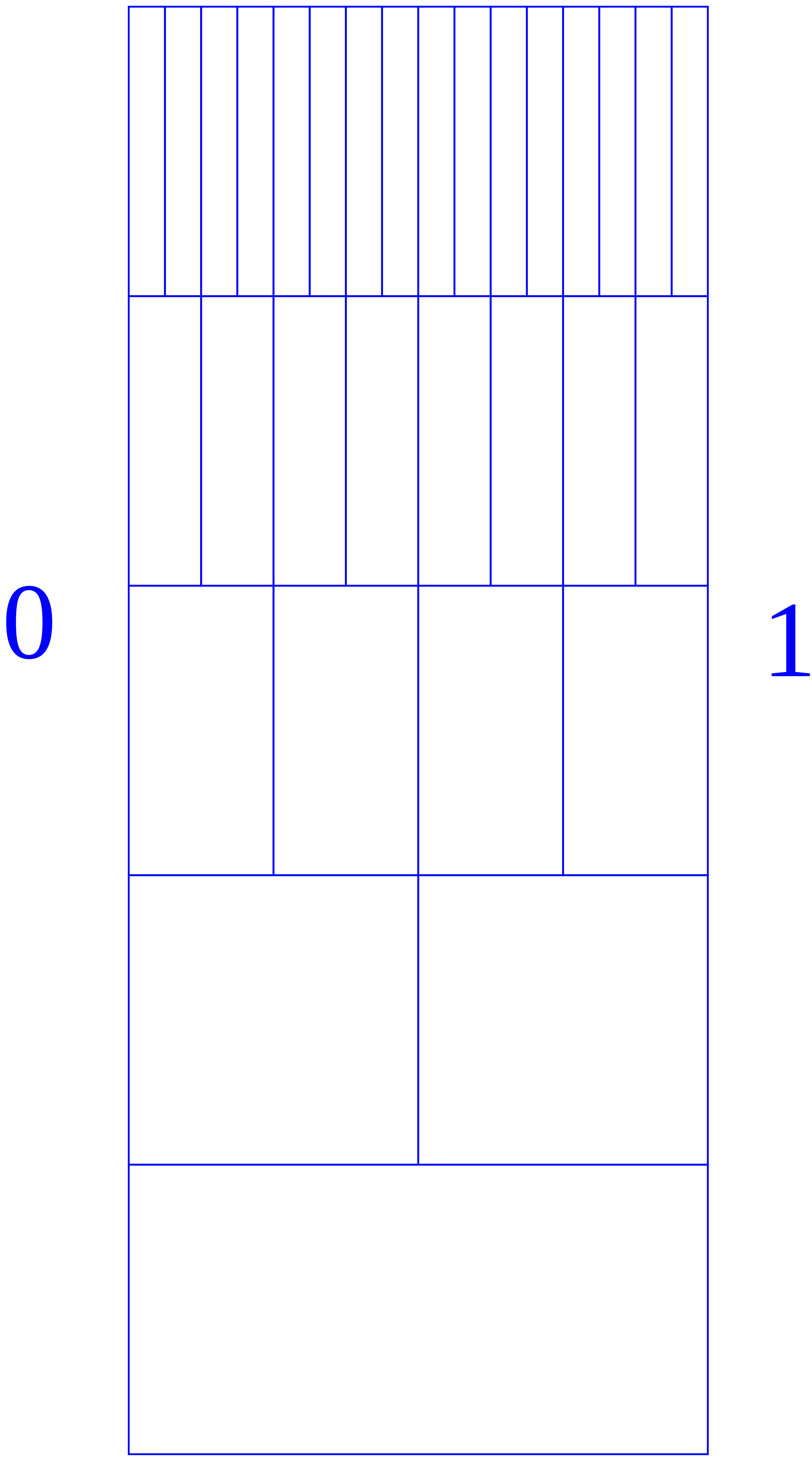
Games without values







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Symmetric, reversible games

THEOREM: If the payoff function F, defined on a subset T of the vertices of an **undirected graph** is bounded between two constants, A and B, then there is a function u which is:

1. The value of the game.

2. The unique bounded infinity harmonic function with the given boundary values.

3. The unique bounded AMLE of F.

THREE MAIN STEPS OF THE PROOF:

1. **Existence** of a bounded infinity harmonic function u.

2. **Standard u-based strategy** implies the connection to AMLE.

3. **Payoff of u achievable for either player**, i.e., given any bounded infinity harmonic u, $V_1 \geq u$ and $V_2 \geq -u$.

From this, we conclude that the value function $V = V_1 = V_2$ exists, and it is the unique bounded infinity harmonic function.

1. Existence

Define u_n to be the best player one can do in a game modified so that if the boundary is not reached in n steps, player one gets A (the lowest possible value). Observe that $u_0(x) = A$ on non-terminal states and

$$u_n(x) = \frac{1}{2}\left(\sup_{y \sim x} u_{n-1}(y) + \inf_{y \sim x} u_{n-1}(y)\right)$$

The u_n's are increasing and bounded between A and B. By induction, each u_n is infinity **sub-harmonic** and the supremum u is clearly infinity **superharmonic** (otherwise it would get bigger after another step), so u is **infinity harmonic**.

Clearly, $V_1 \geq u$, and since player two can play in such a way that u is a supermartingale, $V_1 \leq u$. Hence $u = V_1$.

2. Increasing increment sizes and AMLE

Suppose graph is locally finite and u is bounded and infinity harmonic and players play the **natural strategy suggested by** u, i.e., player 1 always moves to where u is maximal, player 2 to where u is minimal.

If both players play this way and x_n is game position after n steps, $u(x_n)$ is a martingale with **non-decreasing increment sizes**, i.e.,

$$|u(x_{n+1}) - u(x_n)| \geq |u(x_n) - u(x_{n-1})|.$$

Thus, for *any* edge $e = (x, y)$ with $u(y) - u(x) = \delta > 0$ and *any* induced subgraph X' of X containing e, there is a path from y to the boundary of $\partial X'$ on which u **increases by at least** δ **at each step**, and path from x to $\partial X'$ on which u **decreases by at least** δ **at each step**. Conclusion: Lipschitz norm of u in X is at most the Lipschitz norm of u in $\partial X'$. Thus u **is an AMLE**.

3. Value is achievable:

Suppose graph is locally finite, x_0 is starting point, and there is a $\delta > 0$ and a y neighboring x_0 with $|y - x_0| \geq \delta$. Let $\mathcal{V}_\delta$ be the collection of all vertices on which u differs by δ or more from its neighbors.

STRATEGY: when player two leaves $\mathcal{V}_\delta$, player one can always "backtrack" until returning to $\mathcal{V}_\delta$. Let v_n be the last vertex of $\mathcal{V}_\delta$ visited during the first n moves; let y_n be the number of surplus turns player two has had since the last visit to $\mathcal{V}_\delta$. Then observe:

$$u(v_n) - \delta y_n$$

is a **submartingale** which at each step goes up by at least δ with probability 1/2. Convergence follows from martingale convergence theorem, and thus the game must end.

Tug of war with running payoffs

If g is fixed, solutions to $\Delta_\infty u = g$ have meaning as the values of games in which player one collects $g(x)$ from player two each time x is visited.

If g is positive some places and negative other places, the game may not have a value. The reason is that it may turn out that neither player has an incentive to end the game—and each player has to "waste" one or more valuable turns in order to force the game to end.

Fixed targets and comparison with cones

Suppose player one begins the game with a "target a single point" strategy. That is, player one picks a fixed point x_0 and a set S of states and at each turn moves in a way that decreases the distance to x_0 by 1—stopping when game position either reaches x_0 or exits S. Playing in this way makes distance to x_0 a supermartingale, and this leads to an inequality. Namely, for any constants $a > 0, b$, *if* $u(x) \geq a\delta(x, x_0) + b$ **on the boundary of** $S \setminus \{x\}$, then $u(x) \geq a\delta(x, x_0) + b$ **throughout** S.

A function satisfying these inequalities and the corresponding inequalities for player two is said to satisfy **comparison with cones**. It is well known and easy to show that **on a length space, satisfying comparison with cones is equivalent to being an AMLE**.

Value for continuum game

Tug of war variant: **player-one-ϵ-target tug of war**.

At each step, player one targets a point y up to ϵ units away. Then with probability 1/2, player one reaches y (or hits the boundary at a place within $B_{2\epsilon}(y)$) and with probability 1/2 the game state moves to a point in $\overline{B_{2\epsilon}(y)}$ of the second player's choice. If the game does not terminate in n steps, player one receives A, the lowest possible payoff. Denote by v_ϵ^n the value function for this game.

OBSERVE: $v_\epsilon = \sup v_\epsilon^n$ is smaller than or equal to any function which is bounded below by A and satisfies comparison with cones. Define w_ϵ using second player and we have:

Any bounded AMLE u satisfies $v_\epsilon \le u \le w_\epsilon$.

Sandwich argument

CLAIM: $|v_\epsilon - u_\epsilon| = O(\epsilon)$ and $|w_\epsilon - u_\epsilon| = O(\epsilon)$ where u_ϵ is value of ordinary ϵ-step tug of war.

The claim implies $w_\epsilon - v_\epsilon = O(\epsilon)$. Since any AMLE u satisfies $v_\epsilon \le u \le w_\epsilon$, letting ϵ go to zero gives uniqueness.

PROOF OF CLAIM: When game position is more than 2ϵ away from the boundary, one way to think of the game is that player one always takes one step, and then with probability $1/2$ player two gets two steps.

Now, suppose every time player two gets one of these two-step strings, player one uses the next step to **backtrack** the latter of player two's moves. Then this reduces the game to ordinary ϵ tug of war, with an error of $O(\epsilon)$ that comes from what happens near the boundary.