Thick Points of the Gaussian Free Field

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The $d-$ dimensional Gaussian free field (GFF) is a natural $d-$ dimensional dimensional time analog of Brownian motion. It places an important role in statistical physics and the theory of random surfaces. This term paper will focus on the case where $d = 2$.

Let $D \subset \mathbb{C}$ be a bounded domain with smooth boundary and $C_0^\infty(D)$ denote the set of smooth functions compactly supported in D. The Dirichlet inner product is defined by $(f, g)_\triangledown = \int_D \triangledown f \cdot \triangledown g dx$. Let $H(D)$ denote the Hilbert space closure of $C_0^\infty(U)$ under $(\cdot, \cdot)_\triangledown$. The continuum Gaussian free field (GFF) on D is defined formally as a random linear combination

$$h = \sum_{j=1}^{\infty} \alpha_j f_j, \tag{1}$$

where f_j are an ordered orthonormal basis for $H(D)$ and α_i are i.i.d. Gaussian variables defined on the canonical probability space $(\Omega = \mathbb{R}^\mathbb{N}, \mathcal{F}, \mu)$. The formal series (1) does not converge in $H(D)$ almost surely but it converges in $\mathcal{L}_a(D)$ for any $a > 0$ if $d = 2$ (Sheffield, 2007). However for any $f = \sum_j \beta_j f_j \in H(D)$ since $\sum_j \beta_j < \infty$, $\sum_j \beta_j \alpha_j$ converges almost surely. Therefore $(h, f)_\triangledown := \sum_j \beta_j \alpha_j$ is almost surely well-defined and is a Gaussian variable with mean zero and variance $(f, f)_\triangledown$. Furthermore the map $f \in H(D) \to (h, f)_\triangledown \in \Omega$ inherits the Dirichlet innor product structure of $H(D)$, that is

$$E[(h, f)_\triangledown (h, g)_\triangledown] = (f, g)_\triangledown. \tag{2}$$

Example. Let D be the unit torus $\mathbb{R}^2/\mathbb{Z}^2$. The eigenvectors $e_k = e^{2\pi i x\cdot k},\ k \in \mathbb{Z}^2$ of the Laplacian are an orthonormal basis for $L^2(D)$. An orthonormal basis for $H(D)$ can then be explicitly written as $f_k = \frac{1}{2\pi|k|} e^{2\pi i x \cdot k}$. For any given $x \in D$ and any fixed ordering of $k \in \mathbb{Z}^2$, the partial sums of $\sum_{j=1}^{k} \alpha_j f_j(x)$ diverges almost surely since the variance of the partial sums are given by $(2\pi)^{-2} \sum |k|^{-2}$. Therefore h is not well defined as a random variable at any given point $x \in D$.

In the two dimensional case $d =$, the Dirichlet inner product is conformally invariant. Therefore from the example above we know that h will not be well-defined at any given $x \in D$ for any bounded domain D. While it is thus impossible to study h as a random variable at any given point, it is possible to study the average behavior of h on certain subsets of D. Let ρ be any measure on D such that $f \to \int_D f d\rho$ is a continuous linear functional on $H(D)$ which is the case iff $\sum |\int_D f_j d\rho|^2 < \infty$. Then by the duality of Hilbert spaces, there is a unique $\rho_0 \in H(D)$ such that $\int_D f d\rho = (\rho_0, f)_\triangledown$ for all $f \in H(D)$. In fact, $\rho_0 = \sum (\int_D f_j d\rho) f_j$ and $\rho = -\Delta \rho_0$. Consequently $(h, \rho_0)_\triangledown = \sum \alpha_j (\int_D f_j d\rho)$ and can be thought of as the average of h over D under measure ρ.

The measure that is particular simple but elegant is the uniform measure on the circle $\partial D(z, r)$ which we denote by $\mu(z, r)$. It is easy to verify that $\sum |\int_D f_j d\mu(z, r)|^2 < \infty$ and therefore $h(z, r) := \sum \alpha_j (\int_D f_j d\mu(z, r))$ is a.s. well defined. Note that for

$0 \le t_0 \le s \le t$,

$$\mathrm{Cov}(h(z,e^{-t}),h(z,e^{-s})) = (-\Delta^{-1}\mu(z,e^{-t}), -\Delta^{-1}\mu(z,e^{-s}))_{\nabla} = \frac{s}{2\pi} + C(z).$$

Hence $\sqrt{2\pi}h(z,e^{-t}) - \sqrt{2\pi}h(z,e^{-t_0})$ has the same mean and covariance as a standard Brownian motion and let us write $B(z,t)$ for $\sqrt{2\pi}h(z,e^{-t}) - \sqrt{2\pi}h(z,e^{-t_0})$.

By the Brownian law of iterated logarithm, for any $z \in D$.

$$\overline{\lim_{t\to\infty}} \frac{B(z,t)}{\sqrt{2t\log_2 t}} = 1, \ a.s. \tag{3}$$

A natural question to ask is what we can say about $\overline{\lim_{t\to\infty}} \sup_{z\in D} \frac{B(z,t)}{\sqrt{2t\log_2 t}}$. Hu, Miller and Peres (2009) defined $T^C(a;D) = \{z \in D : \lim_{t\to\infty} \frac{B(z,t)}{\sqrt{2t}} = \sqrt{a}\}$ and proved the following theorem:

Theorem 1. *(Hu, Miller and Peres) The Hausdorff dimension of $T^c(a;D)$ is almost surely $2-a$ for any $0 \le a \le 2$. If $a > 2$, $T^c(a;D)$ is almost surely empty.*

They proved the theorem in two steps. First they showed that $T^C_{\ge}(a;D) := \{z \in D : \limsup_{t\to\infty} \frac{B(z,t)}{\sqrt{2t}} \ge \sqrt{a}\}$ has Hausdorff dimension at most $2-a$ for $0 \le a \le 2$ and $T^C_{\ge}(a;D)$ is empty a.s. if $a > 2$. Second they showed $\dim_H T^C(a;D) \ge 2-a$.

The key in their argument for the first conclusion is the fact that $h(z,r)$ has a locally γ-Hölder continuous modification if $\gamma < 1/2$. More specifically, the following was proved by Hu, Miller and Peres (2009).

Proposition 2. *(Hu, Miller and Peres) The circle average $h(z,r)$ has a modification $\hat{h}(z,r)$ such that for any $0 < \gamma < 1/2$ and $\varepsilon, \xi > 0$ there exists $M = M(\gamma,\varepsilon,\xi)$ such that*

$$|\hat{h}(z,r) - \hat{h}(w,s)| \le M(\log\frac{1}{r})^{\xi} \frac{|(z,r)-(w,s)|^{\gamma}}{r^{\gamma+\varepsilon}} \tag{4}$$

for all $z,w \in D$ and $r,s \in (0,1]$ with $1/2 \le r/2 \le 2$.

With Proposition (2), the authors showed $|B(z,t) - B(z,K\log n)| \le O((\log n)^{\xi})$ for any $\xi < 1$ and $K\log n < t < K\log(n+1)$ thus reducing the problem to discrete time points. Second $|B(z,K\log n) - B(z_{nj},K\log n)| \le O((\log n)^{\xi})$ where (z_{nj}) is a maximal n^{-K} net of D and $z \in D(z_{nj},n^{-K})$. Then they tried to show that the following set contains $T^C_{\ge}(a;D)$ for any large N:

$$I(a,N) = \bigcup_{n\ge N} \{D(z_{nj},n^{-K}) : j \in I_n\} \tag{5}$$

where

$$I_n = \{j : \frac{|B(z_{nj},K\log n)|}{\sqrt{2}K\log n} \ge \sqrt{a} - C(\log n)^{\xi-1}\} \tag{6}$$

A classic inequality will give

$$P(\frac{|B(z_{nj},K\log n)|}{\sqrt{2}K\log n} \ge \sqrt{a} - C(\log n)^{\xi-1}) = O(n^{-Ka-o(1)})$$

which leads to a bound on $E|I_n|$ and $E[\sum_{n\ge N}\sum_{j\in I_n} \mathrm{diam} D(z_{nj},n^{-K})^{\alpha}] \to 0 \ N \to \infty$ for any $\alpha = 2-a+(2+a)/K$. For $a > 2$, $E|I_n| \to 0$.

While the basic idea is clear, the conclusion made at (5) needs more justification. For example, if $B(z, K\log n)/(\sqrt{2}K\log n) = \sqrt{a} - \sqrt{(\log n)^{\xi-1}}$, then we have $\limsup_{t\to\infty} \frac{B(z,t)}{\sqrt{2}t} \geq \sqrt{a}$ but we cannot conclude $j \in I_n$ just knowing $z \in D(z_{nj}, n^{-K})$.

The lower bound $T^C(a; D) \geq 2 - a$ is more involved. It calls for a result (Theorem 8.7) from Martin (1995). The α-energy of a measure τ on D is defined as

$$I_\alpha(\tau) = \int\int |x-y|^{-\alpha} d\tau(x)d\tau(y). \tag{7}$$

Theorem 8.7 of Martin (1995) implies that if $I_\alpha(\tau) < \infty$, then the support of τ has Hausdorff dimension at least α. Hu, Miller and Peres considered measures τ_n concentrated in the neighborhoods of a finite subset of what is called n-perfect a-thick points. The set of n-perfect a-thick points is $E^n = \{z : |B(z,t) - B(z,t_m) - \sqrt{2a}(t - t_m)| \leq \sqrt{t_{m+1} - t_m},\ \forall m \leq n\}$. Note that $B(z,t) - B(z,t_m)$ is defined on the annulus $D(z, e^{-t_m})/D(z, e^{-t})$ and for different m the annuli are disjoint. The Markov property of GFF implies $B(z,t) - B(z,t_m),\ t_m < t < t_{m+1}$ and $B(z,t) - B(z,t_n),\ t_n < t < t_{n+1}$ are disjoint. This allows them to get the following estimate:

$$P(z, w \in E^n) \leq O(|z-w|^{-a-\varepsilon})P(z \in E^n)P(w \in E^n) \tag{8}$$

for all large n and any $\varepsilon > 0$. This joint probability estimate made it possible to show that $EI_{2-a-\varepsilon}(\tau_n) < B < \infty,\ \forall n$. (8) also implies that $\tau_n(D)$ has uniformly bounded first and second moments. Consequently by Paley-Zygmund inequality there exists $b, d, v > 0$ such that $G_n = \{b \leq \tau_n(D) \leq b^{-1}, I_{2-a-\varepsilon}(\tau_n) \leq d\}$ has probability measure $P(G_n) > v$ and thus $P(G) > 0$ for $G = \limsup_n G_n$. For any $w \in G$, the lower semi-continuity of I_α implies that there is measure τ with $b \leq \tau(D) \leq b^{-1}, I_{2-a-\varepsilon}(\tau) \leq d$ that concentrates on $P_a(w)$ where P_a is the set of points contained in the support of τ_n for infinitely many n and thus measurable. Therefore $\dim_H P_a(w) \geq 2-a-\varepsilon$ for every $w \in G$. Then Hewitt-Savage zero-one law implies that $P(\dim_H P_a(w) \geq 2-a-\varepsilon) = 1$

It is worth mentioning that Xu, Miller and Peres originally defined $z \in D$ to be an a-thick point if

$$\lim_{r\to 0} \frac{\int_{D(z,r)} h(x)dx}{\pi r^2 \log\frac{1}{r}} = \sqrt{\frac{a}{\pi}}. \tag{9}$$

Since $1_{D(z,r)} \in \mathcal{L}_b(D)$ for $-1/2 < b < 0$, the dual pairing of $1_{D(z,r)}$ and h implies that $\int_{D(z,r)} h(x)dx$ is continuous in (z,r) while by Proposition 2 $h(z,s)$ has a continuous modification. Therefore it is not hard to see that almost surely

$$\int_0^r 2\pi s h(z,s)ds = \int_{D(z,r)} h(x)dx, \text{ for all } z.$$

From this equality they obtained the collection of thick points $T^C(a; D)$. Theorem 1 thus translates to the following:

Theorem 3. *(Hu, Miller and Peres) Let $T(a,D)$ denote the set of a-thick points. The Hausdorff dimension of $T(a,D)$ is almost surely $2-a$ for any $0 \leq a \leq 2$. If $a > 2$, $T(a,D)$ is almost surely empty.*

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