18.440 PROBLEM SET FOUR, DUE MARCH 7

A. FROM TEXTBOOK CHAPTER FOUR:

1. Problem 23: You have \$1000, and a certain commodity presently sells \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely.
   (a) If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?
   (b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?
2. Problem 35: A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win -\$1.00. (That is, you lose \$1.00.) Calculate
   (a) the expected value of the amount you win;
   (b) the variance of the amount you win.
3. Problem 50: Suppose that a biased coin that lands on heads with probability p is flipped 10 times. Given that a total of 6 heads results, find the conditional probability that the first 3 outcomes are
   (a) h, t, t (meaning that the first flip results in heads, the second in tails, and the third in tails);
   (b) t, h, t.
4. Problem 57: The probability of being dealt a full house in a hand of poker is approximately .0014. Find an approximation for the probability that, in 1000 hands of poker, you will be dealt at least 2 full houses.
5. Theoretical Exercise 13: Let X be a binomial random variable with parameters (n, p). What value of p maximizes $P\{X = k\}, k = 0, 1, 2, \dots, n$? This is an example of a statistical method used to estimate p when a binomial (n, p) random variable is observed to equal k. If we assume that n is known, then we estimate

p by choosing that value of p which maximizes $P\{X = k\}$. This is known as the *method of maximum likelihood estimation.*

6. Theoretical Exercise 19: Show that if X is a Poisson random variable with parameter λ, then

$$E[X^n] = \lambda E[(X+1)^{n-1}].$$

Now use this result to compute $E[X^3]$.

B. Define the covariance $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$.

1. Check that $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$, that $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$, and that $\mathrm{Cov}(\cdot,\cdot)$ is a bilinear function of its arguments. That is, if one fixes one argument then it is a linear function of the other. For example, if we fix the second argument then for real constants a and b we have $\mathrm{Cov}(aX + bY, Z) = a\mathrm{Cov}(X,Z) + b\mathrm{Cov}(Y,Z)$.
2. If $\mathrm{Cov}(X_i, X_j) = ij$, find $\mathrm{Cov}(X_1 - X_2, X_3 - 2X_4)$.
3. If $\mathrm{Cov}(X_i, X_j) = ij$, find $\mathrm{Var}(X_1 + 2X_2 + 3X_3)$.

C. Instead of maximizing her expected wealth $E[W]$, Jill maximizes $E[U(W)]$ where $U(x) = -(x - x_0)^2$ and x_0 is a large positive number. That is, Jill has a *quadratic utility function.* (It may seem odd that Jill's utility declines with wealth once wealth exceeds x_0. Let us assume x_0 is large enough so that this is unlikely.) Jill currently has W_0 dollars. You propose to sample a random variable X (with mean μ and variance σ^2) and to give her X dollars (she will lose money if X is negative) so that her new wealth becomes $W = W_0 + X$.

1. Show that $E[U(W)]$ depends on μ and σ^2 (but not on any other information about the probability distribution of X) and compute $E[U(W)]$ as a function of x_0, W_0, μ, σ^2.
2. Show that given μ, Jill would prefer for σ^2 to be as small as possible. (One sometimes refers to σ as *risk* and says that Jill is *risk averse.*)
3. Suppose that $X = \sum_{i=1}^{n} a_i X_i$ where a_i are fixed constants and the X_i are random variables with $E[X_i] = \mu_i$ and $\mathrm{Cov}[X_i, X_j] = \sigma_{ij}$. Show that in this case $E[U(W)]$ depends on the μ_i and the σ_{ij} (but not on any other information about the joint probability distributions of the X_i) and compute $E[U(W)]$. Hint: first compute the mean and variance of X.

4. Read the Wikipedia article on “Modern Portfolio Theory”. Summarize what you learned in two or three sentences.