

**18.440 PROBLEM SET THREE, DUE FEBRUARY 28**

A. FROM TEXTBOOK CHAPTER THREE:

1. Problem 26: Suppose that 5 percent of men and .25 percent of women are color blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females. What if the population consisted of twice as many males as females?
2. Problem 43: There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the 3 coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?
3. Problem 47: An urn contains 5 white and 10 black balls. A fair die is rolled and that number of balls is randomly chosen from the urn. What is the probability that all of the balls selected are white? What is the conditional probability that the die landed on 3 if all the balls selected are white?
4. Suppose that  $E$  and  $F$  are mutually exclusive events of an experiment. Show that if independent trials of this experiment are performed, then  $E$  will occur before  $F$  with probability  $P(E)/[P(E) + P(F)]$ .
5. Theoretical Exercise 1. Show that if  $P(A) > 0$ , then

$$P(AB|A) \geq P(AB|A \cup B).$$

6. Theoretical Exercise 24: A round-robin tournament of  $n$  contestants is a tournament in which each of the  $\binom{n}{2}$  pairs of contestants play each other exactly once, with the outcome of any play being that one of the contestants wins and the other loses. For a fixed integer  $k$ ,  $k < n$ , a question of interest is whether it is possible that the tournament outcome is such that, for every set of  $k$  players, there is a player who beat each member of that set. Show that if

$$\binom{n}{k} \left[1 - \left(\frac{1}{2}\right)^k\right]^{n-k} < 1$$

then such an outcome is possible. *Hint:* Suppose that the results of the games are independent and that each game is equally likely to be

won by either contestant. Number the  $\binom{n}{k}$  sets of  $k$  contestants, and let  $B_i$  denote the event that no contestant beat all of the  $k$  players in the  $i$ th set. Then use Boole's inequality to bound  $P(\cup_i B_i)$ .

B. Suppose that a fair coin is tossed infinitely many times, independently. Let  $X_i$  denote the outcome of the  $i$ th coin toss (an element of  $\{H, T\}$ ). Compute the probability that:

1.  $X_i = H$  for all positive integers  $i$ .
2. The pattern HHTTHHTT occurs at some point in the sequence  $X_1, X_2, X_3, \dots$

C. Two unfair dice are tossed. Let  $p_{i,j}$ , for  $i$  and  $j$  in  $\{1, 2, 3, 4, 5, 6\}$ , denote the probability that the first die comes up  $i$  and the second  $j$ . Suppose that for any  $i$  and  $j$  in  $\{1, 2, 3, 4, 5, 6\}$  the event that the first die comes up  $i$  is independent of the event that the second die comes up  $j$ . Show that this independence implies that, as a 6 by 6 matrix,  $p_{i,j}$  has rank one (i.e., show that there is some column of the matrix such that each of the other five column vectors is a constant multiple of that one).