SCHRAMM-LOEWNER EVOLUTION

and the

GAUSSIAN FREE FIELD

Scott Sheffield

includes joint work with Schramm; Schramm and Wilson; and Werner

The *standard Gaussian* on n-dimensional Hilbert space

has density function $e^{-(v,v)/2}$ (times an appropriate constant). We can write a sample from this distribution as

$$\sum_{i=1}^{n} \alpha_i v_i$$

where the v_i are an orthonormal basis for $\mathbb{R}^n$ under the given inner product, and the α_i are mean zero, unit variance Gaussians.

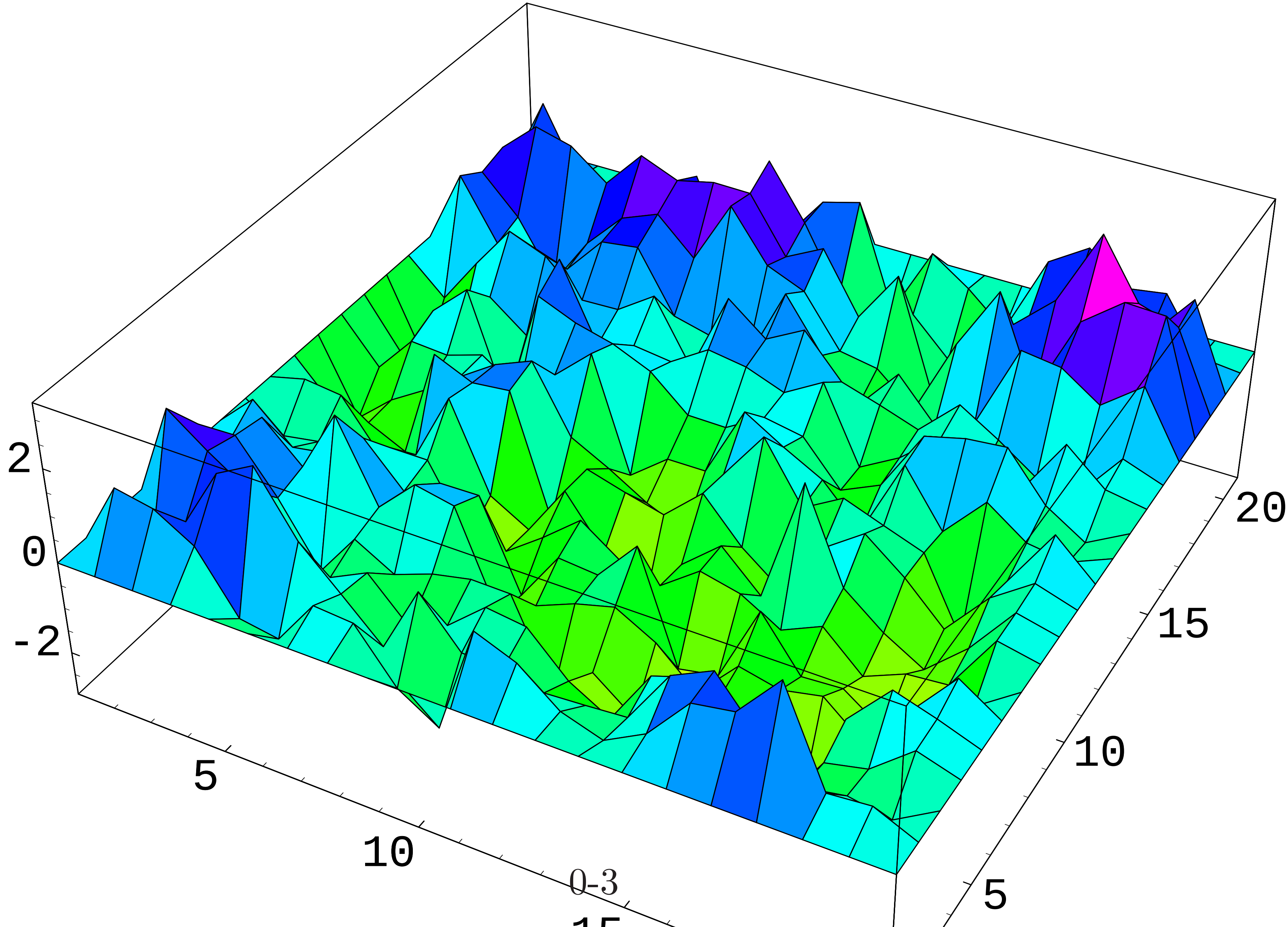
The discrete Gaussian free field

The **Dirichlet energy** of a real function f on the vertices of a planar graph Λ is $H(f) = (f, f)_\nabla$ where $(f, g)_\nabla$ is the **Dirichlet form**

$$(f, g)_\nabla = \sum_{x \sim y} (f(x) - f(y))\,(g(x) - g(y))\,.$$

Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



Some DGFF properties:

Zero boundary conditions: The Dirichlet form $(f, f)_\nabla$ is an inner product on the space of functions with zero boundary, and the DGFF is a standard Gaussian on this space.

Other boundary conditions: DGFF with boundary conditions f_0 is an affine translation of DGFF with zero boundary; i.e., the same as DGFF with zero boundary conditions *plus* the (discrete) harmonic interpolation of f_0 to Λ.

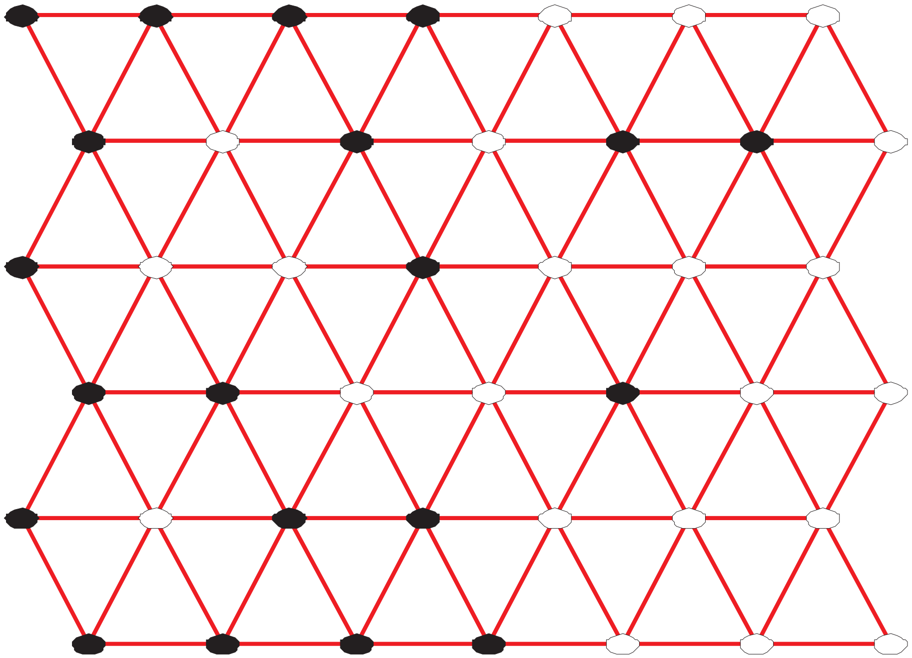
Markov property: **Given** the values of f on the boundary of a subgraph Λ' of Λ, the values of f on the remainder of Λ' have the law of a DGFF on Λ', with boundary condition given by the observed values of f on $\partial\Lambda'$.

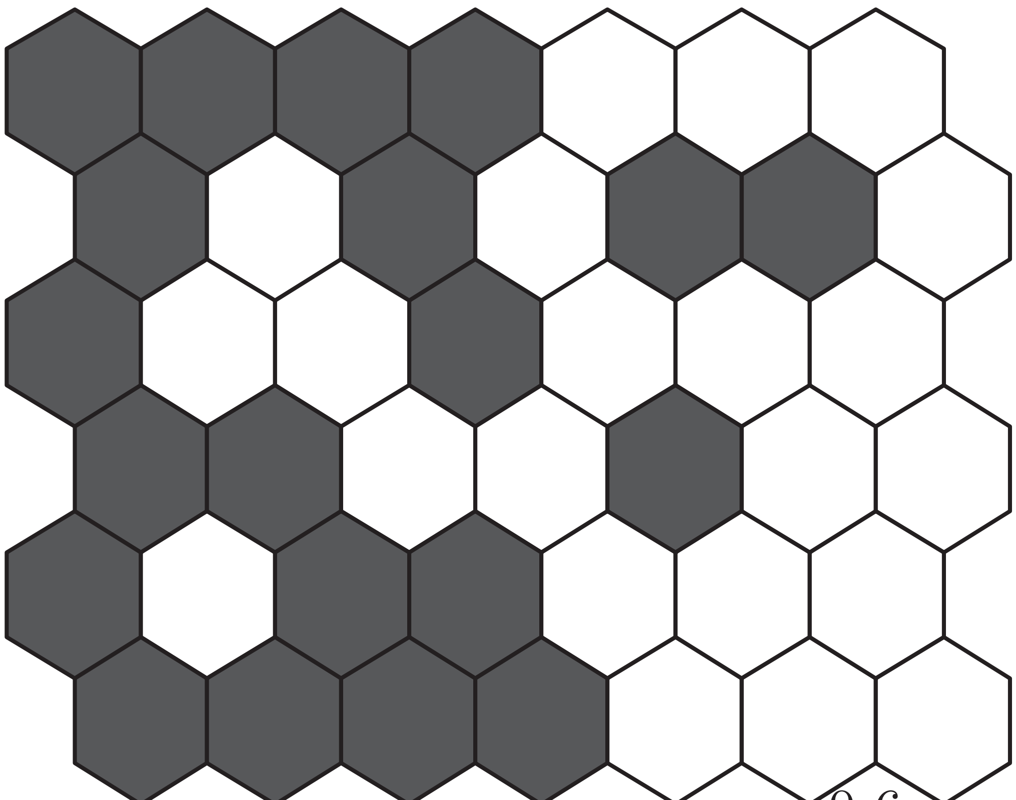
The continuum Gaussian free field

is a “standard Gaussian” on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx dy.$$

One way to view GFF: A formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—inner products (h, ϕ) are well defined whenever ϕ is sufficiently smooth. The projection of the GFF onto the space of functions piecewise linear on triangle lattice triangles gives the DGFF (times the lattice-dependent constant $3^{1/4}$).





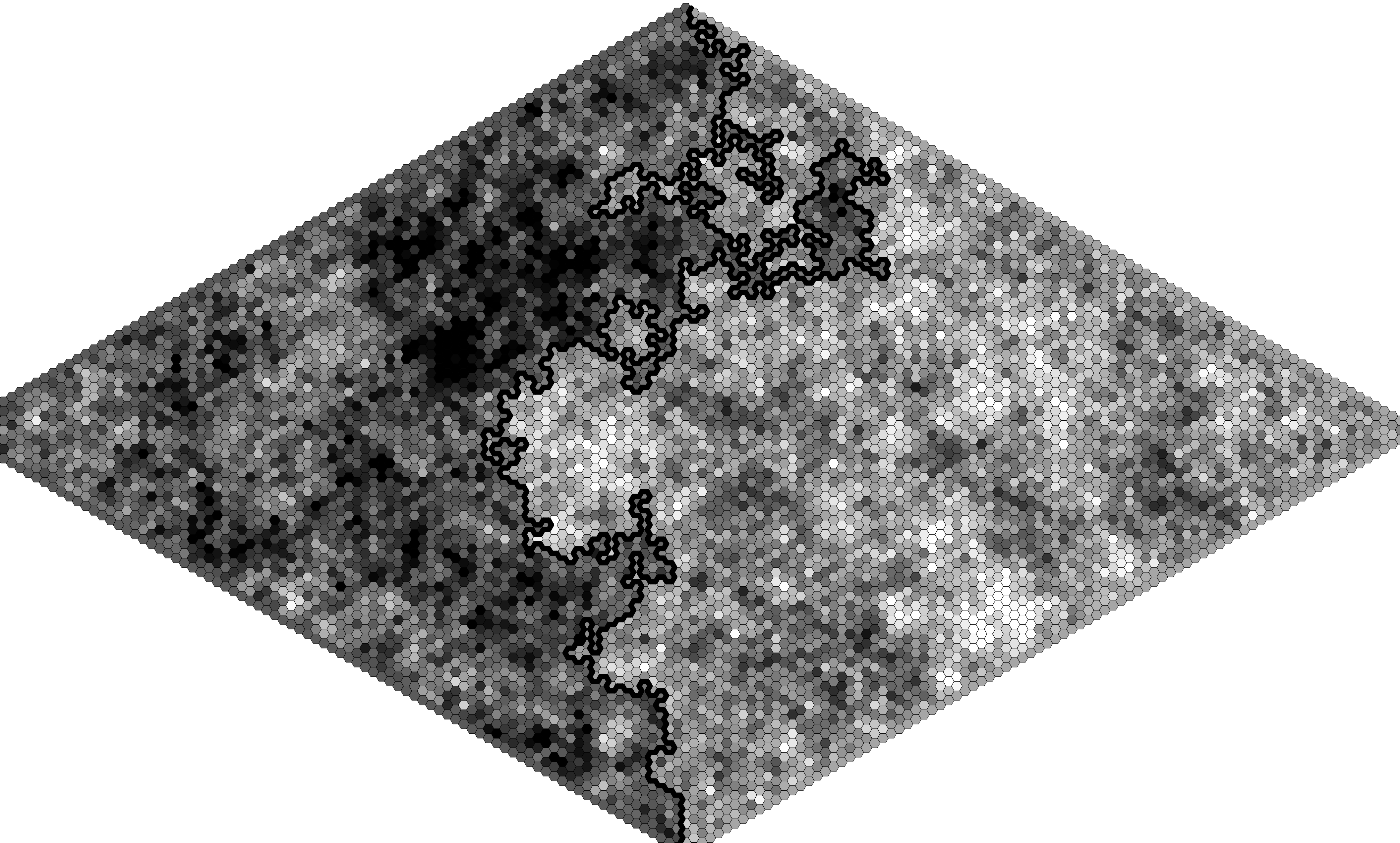
0-6

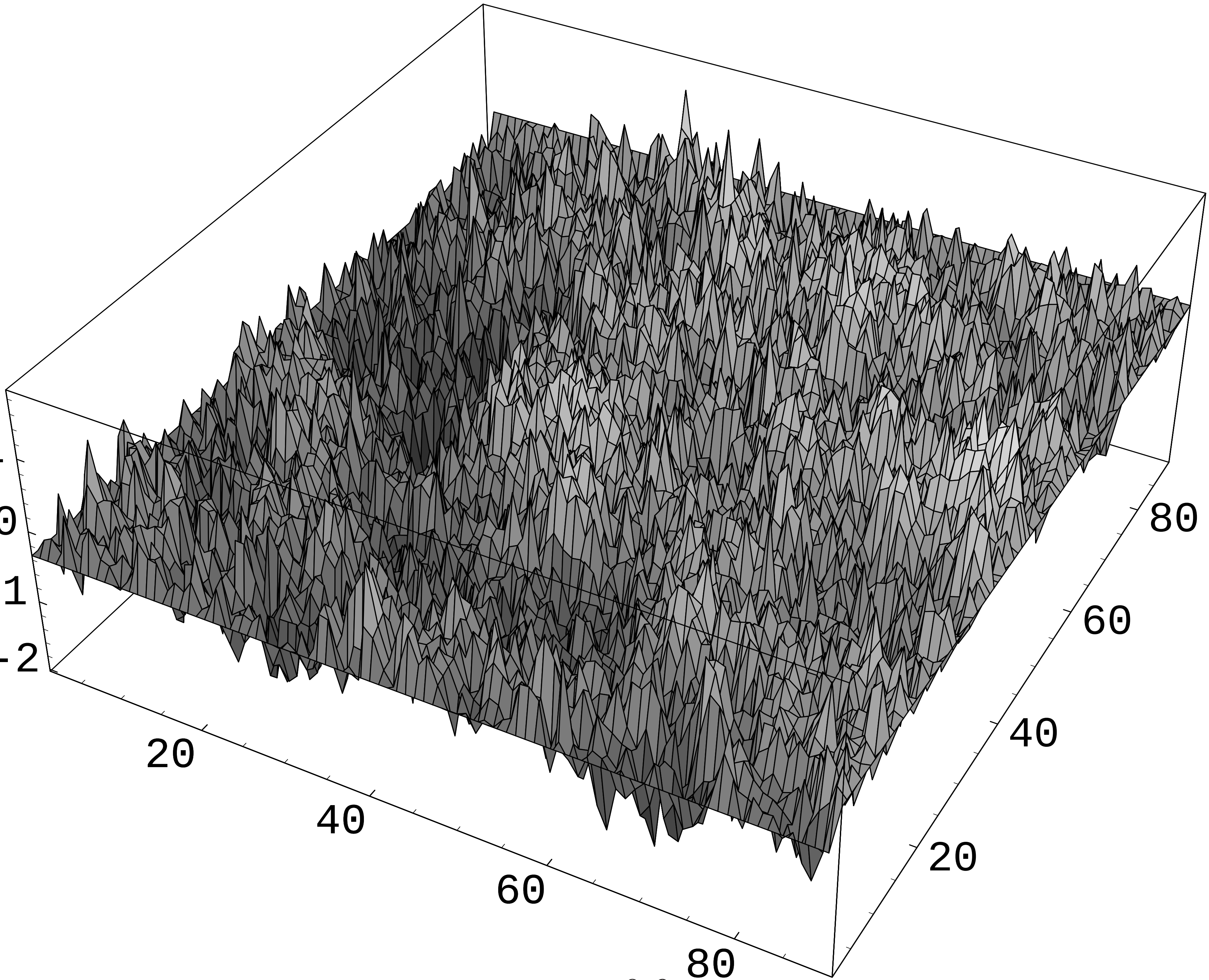
Scaling limit of zero-height contour line

Theorem (Schramm, S): If initial boundary heights are λ on one boundary arc and $-\lambda$ on the complementary arc, where λ is the constant $\sqrt{\frac{\pi}{8}}$, then the scaling limit of the zero-height interface (as the mesh size tends to zero) is SLE_4.

If the initial boundary heights are instead are instead $-(1+a)\lambda$ and $(1+b)\lambda$, then as the mesh gets finer, the laws of the random paths described above converge to the law of $\mathrm{SLE}_{4,a,b}$.

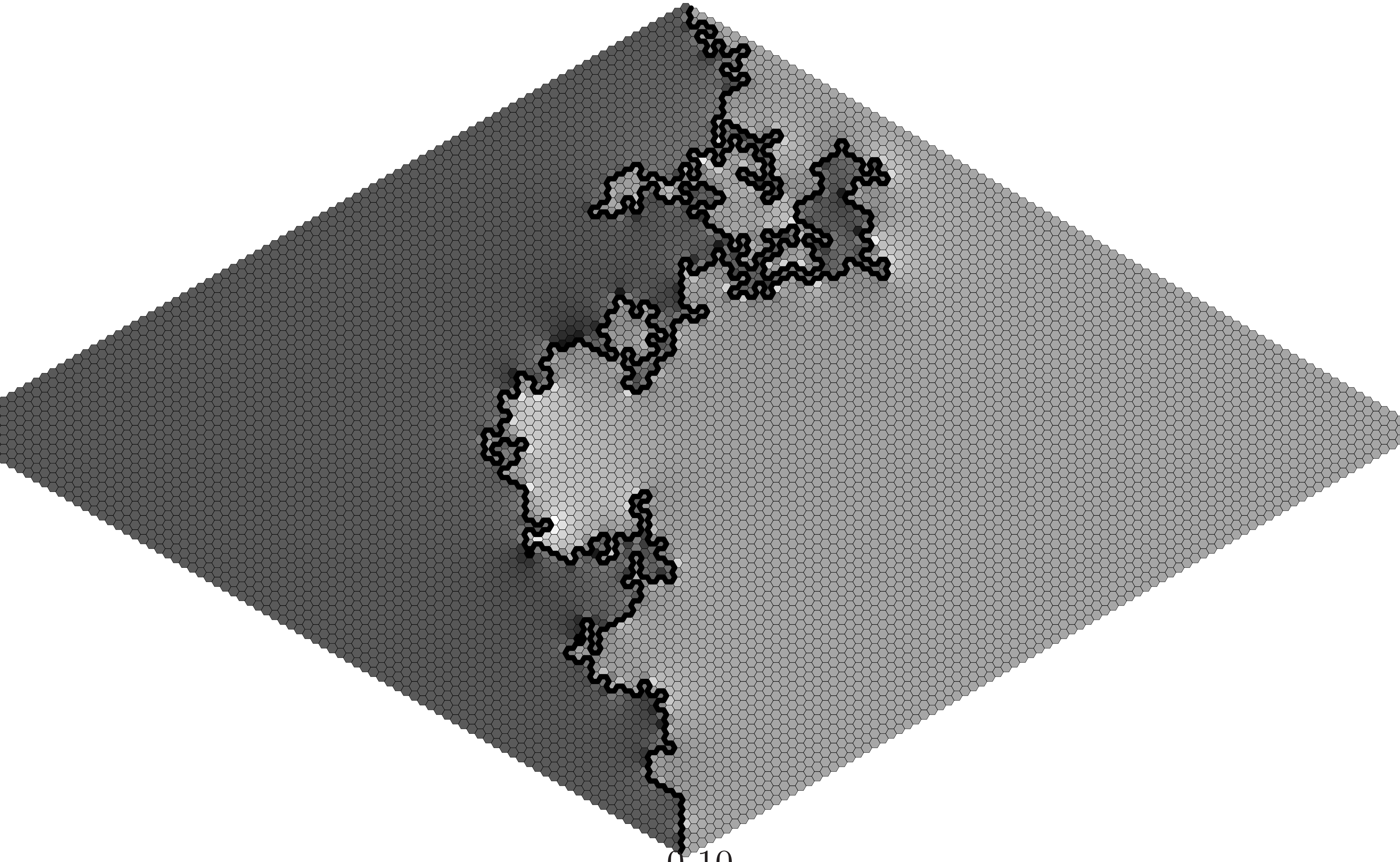
DGFF with $\pm\lambda$ boundary conditions



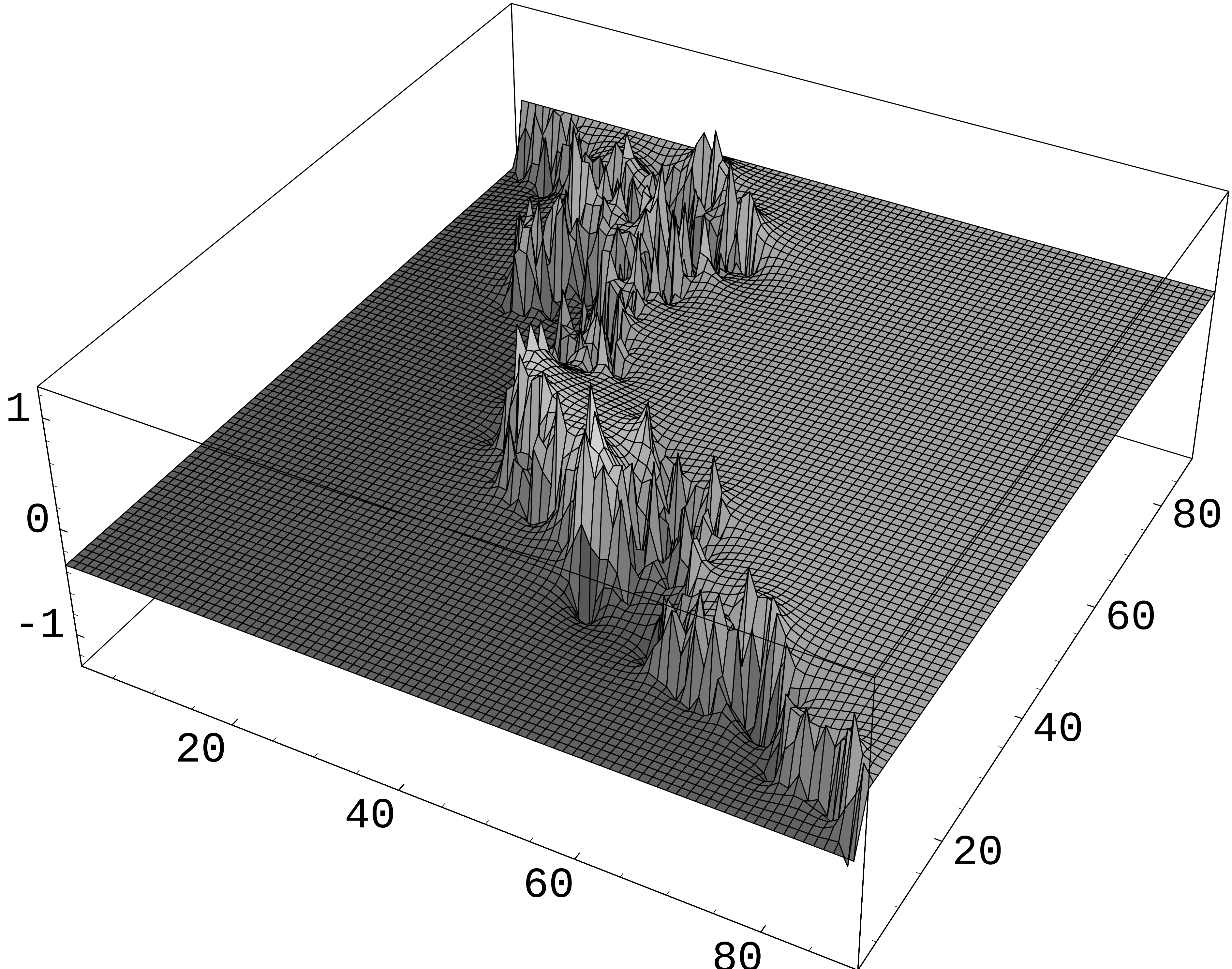


1
0
-1
-2
20
40
60
80
20
40
60
80

Expectations given values along interface

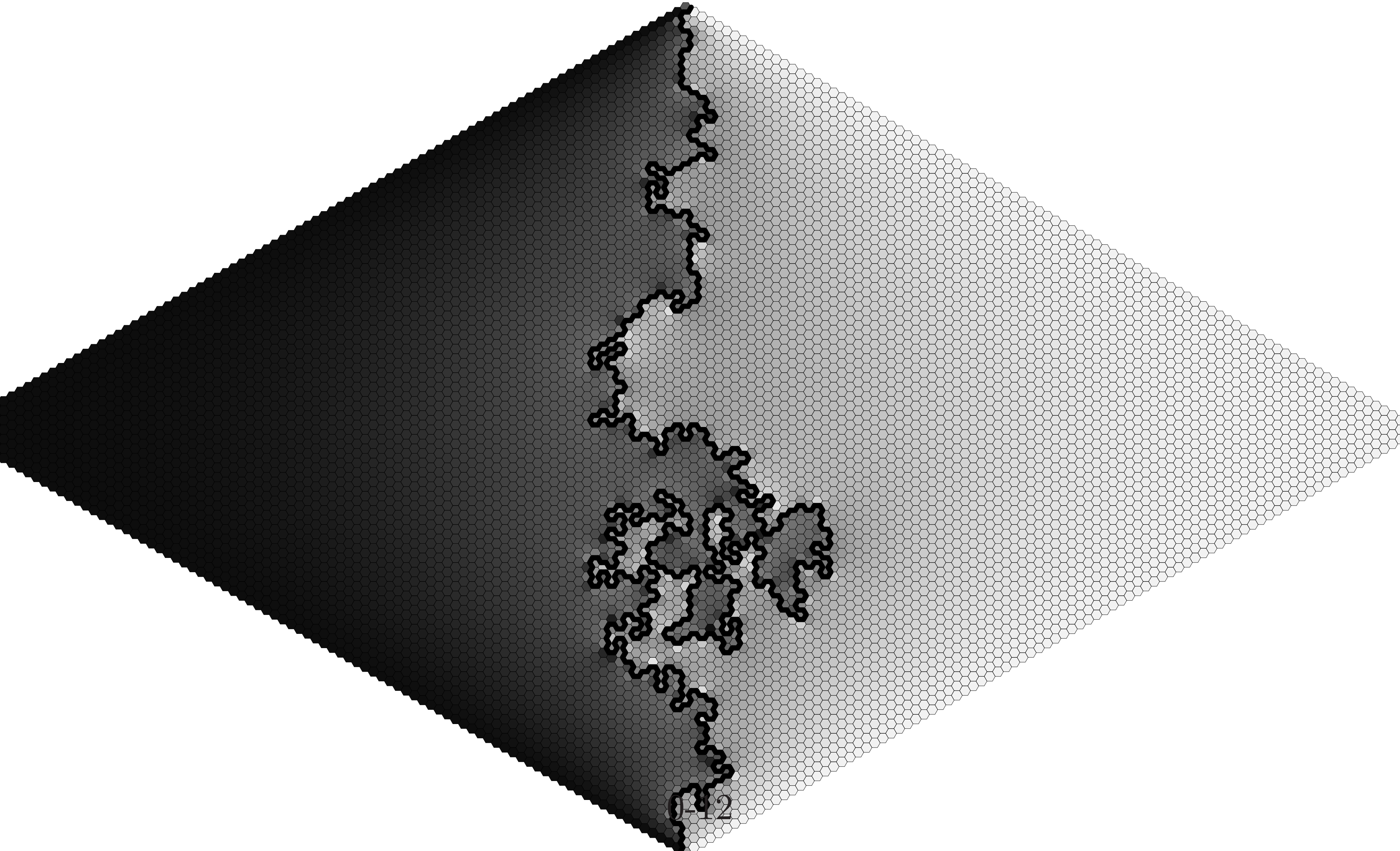


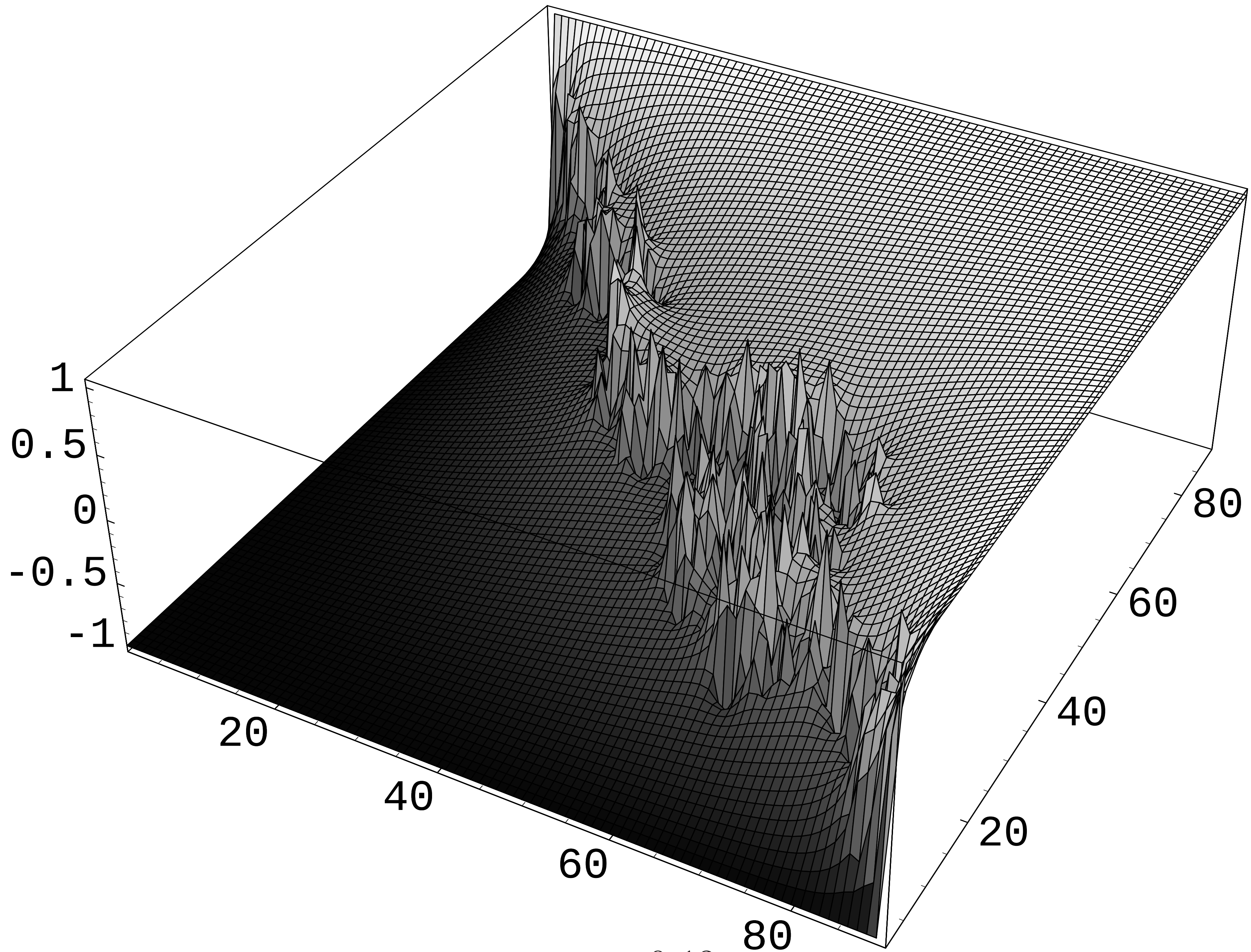
0-10



1
0
-1
20
40
60
80
20
40
60
80

Expectations given interface, $\pm 3\lambda$ boundary conditions



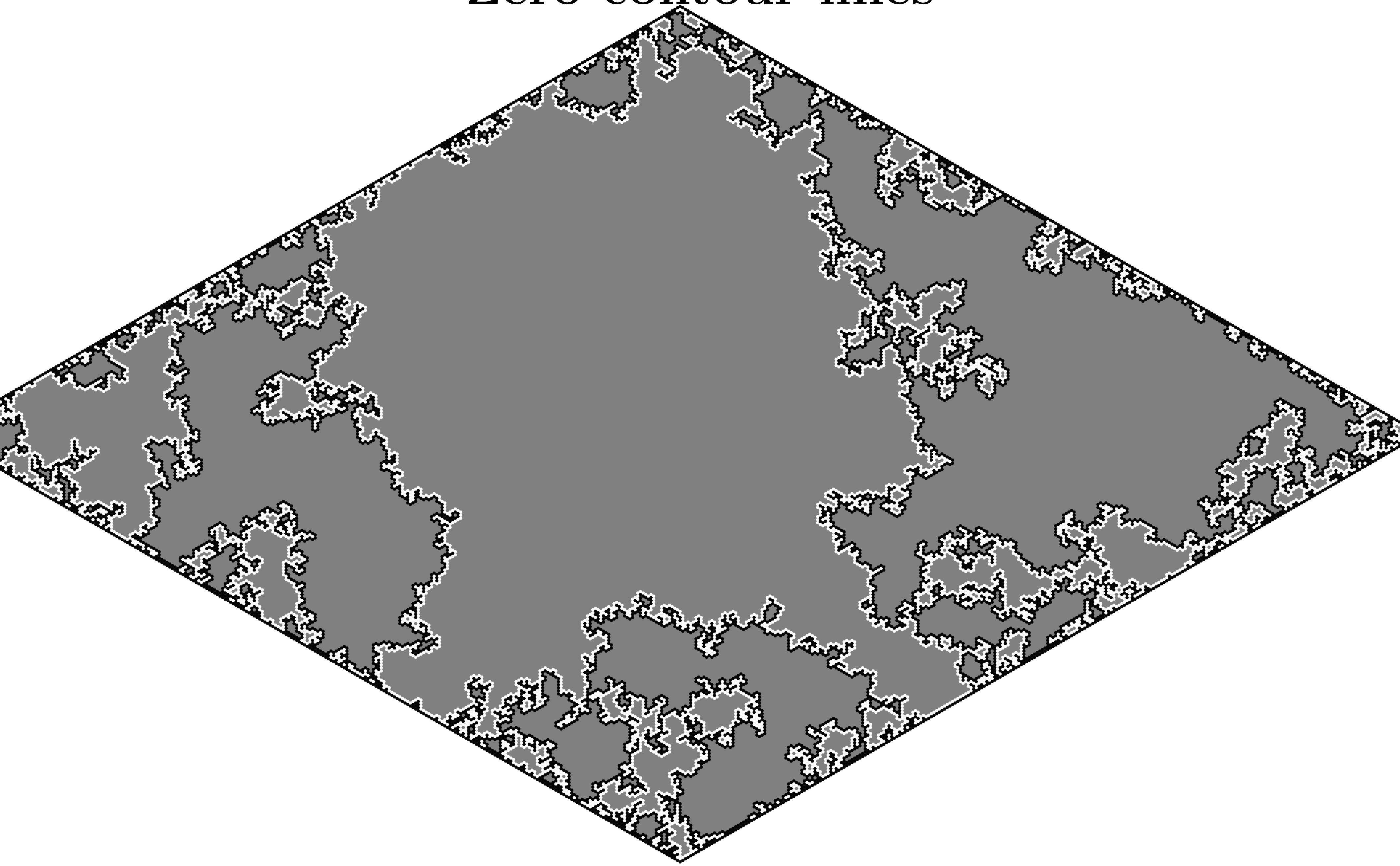


1
0.5
0
-0.5
-1
20
40
60
80
20
40
60
80

Main ideas behind the proof

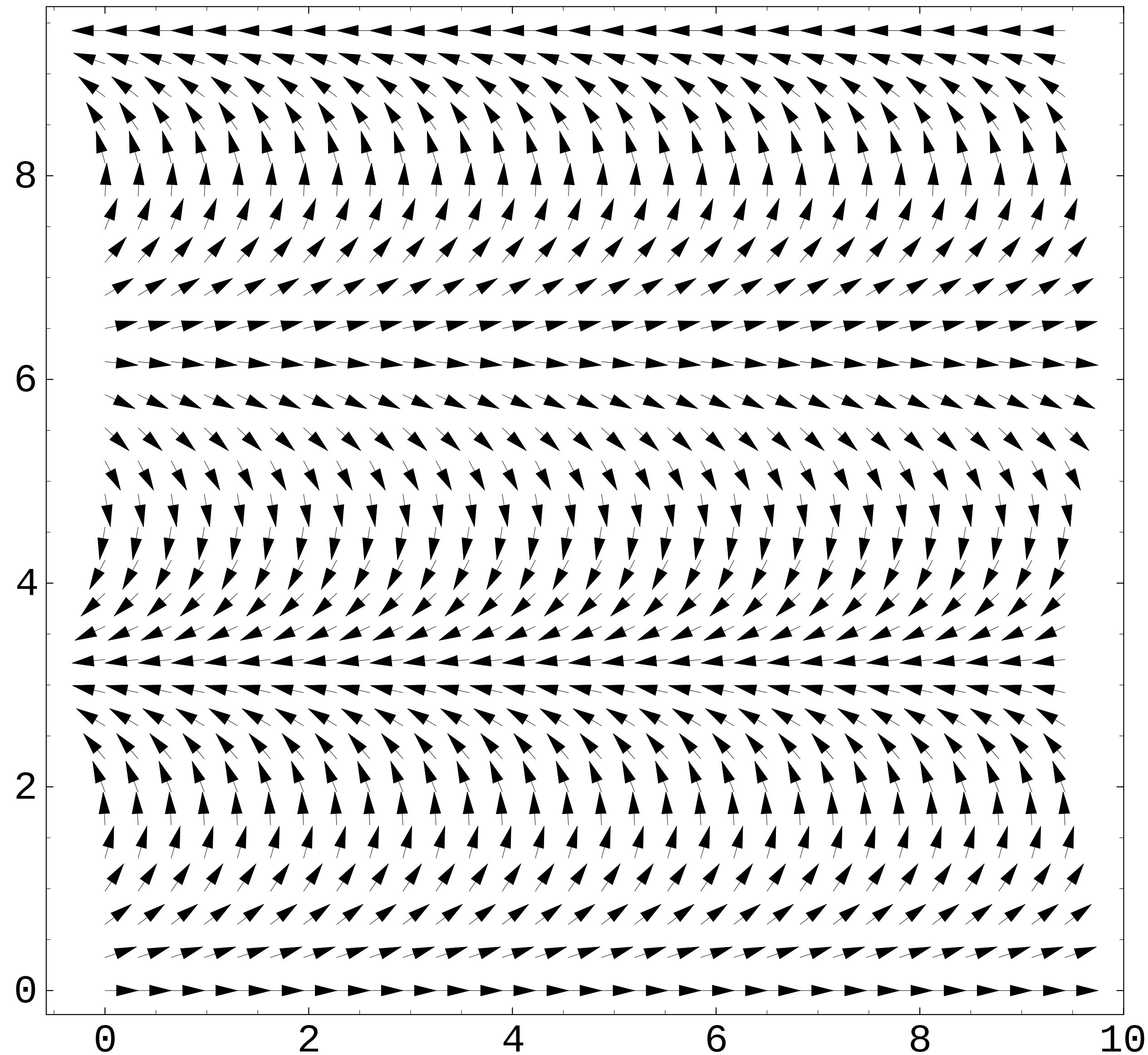
1. Observe: SLE_4 is the only random path γ with the following property: *Given* $\gamma([0,t])$, the probability that γ passes z on right equals the probability that Brownian motion started at z first hits $\mathbb{R} \cap \gamma[0,t]$ on the left side of $\gamma(t)$.
2. Prove: there is a unique (up to redefinition on measure zero set) path-valued function $h \to \gamma^h$ of the GFF with $\pm\lambda$ boundary conditions such that *given* γ^h, the conditional law of h is that of $-\lambda 1_L + \lambda 1_R + h_L + h_R$, where L and R are the subsets of D on the left and right sides of γ and h_L and h_R are independent GFFs on those domains.
3. Prove: given $\gamma^h([0,t])$, the conditional law of h is that of a GFF on $D \backslash \gamma^h([0,t])$ with $\pm\lambda$ boundary. This property determines λ.
4. For each discrete mesh Λ, obtain DGFF by projecting h onto space of piecewise linear functions on triangles of Λ; let γ^h_Λ be the corresponding contour. Prove: the fine-mesh limit of the γ^h_Λ is γ^h.

Zero contour lines

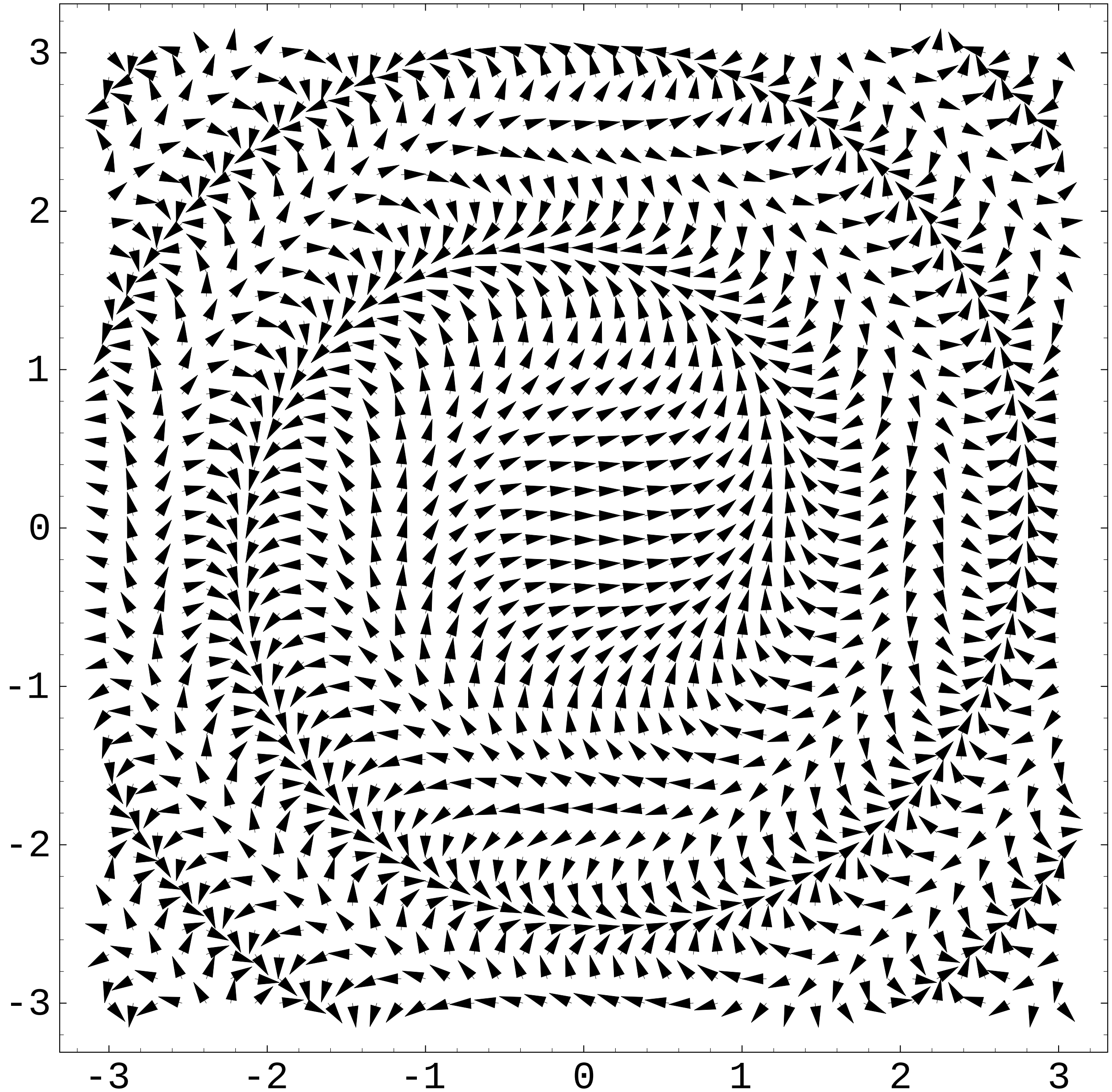


0-15

Vector Field e^{ih} **where** $h(x, y) = \pi/2 - y$



Vector Field e^{ih} where $h(x, y) = x^2 + y^2$



Altimeter compass geometry

A ray in the altimeter compass geometry is a flow line of $e^{2\pi i(\alpha+h/\chi)}$ for some α.
Now let's modify our sense of direction. Call the direction $e^{2\pi i(\alpha+h/\chi)}$

1. East if $\alpha = 0$.
2. North if $\alpha = .25$.
3. West if $\alpha = .5$.
4. South if $\alpha = .75$.

If $h = 0$, then the rays of the AC geometry are those of ordinary Euclidean geometry. More generally, if h is Lipschitz, then the flow line of $e^{2\pi i(\alpha+h/\chi)}$ starting at a given point exists and is uniquely defined.

Conformal maps of AC geometries

Let h be defined on a domain D. Let g be a conformal map from D to D'. Fix $\chi > 0$. Then the AC geometry of (D, h) is the same as that of

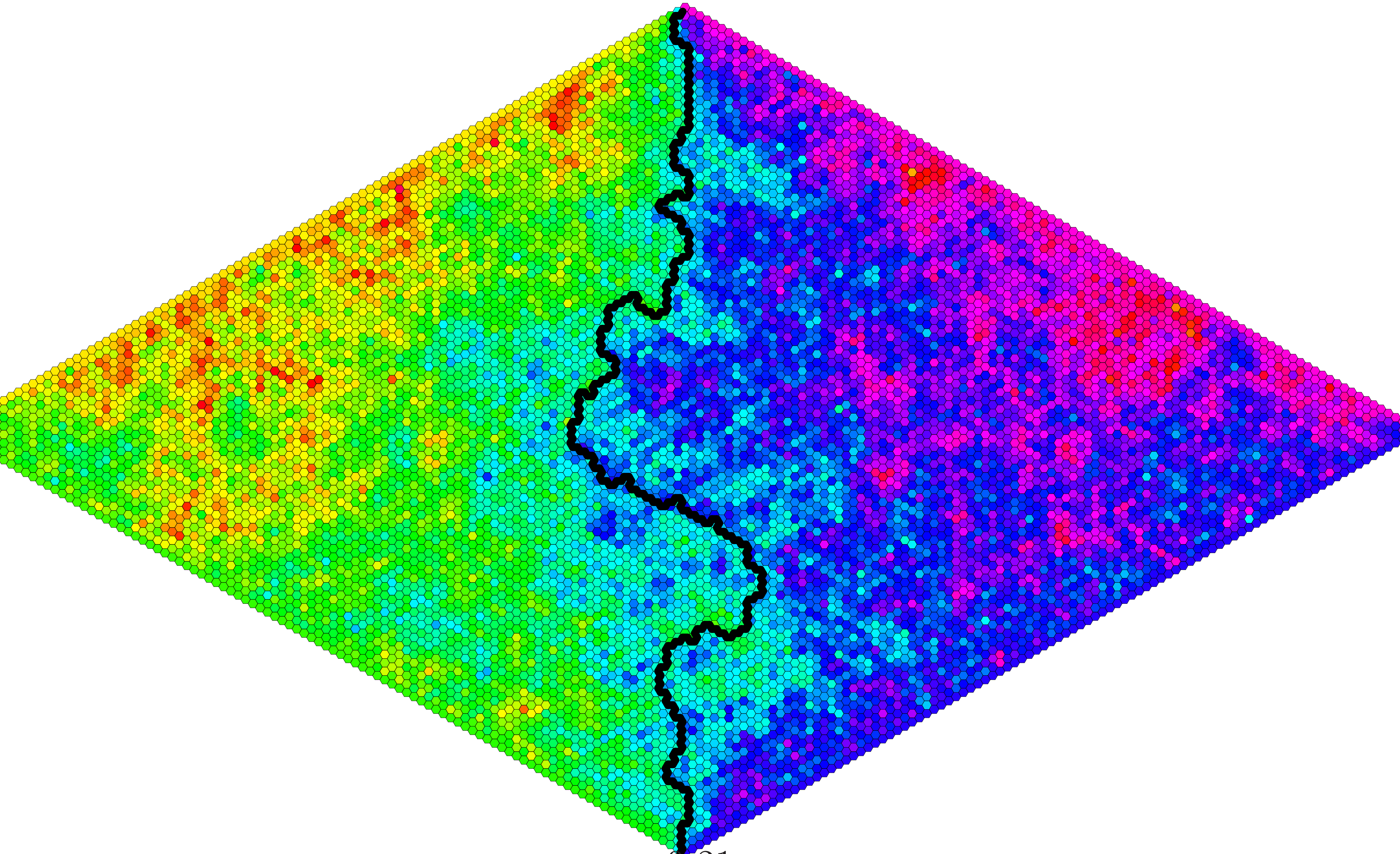
$$(D', h + (\chi/2\pi) \arg g').$$

This implies in particular that if h is harmonic, then the rays are locally the images (under a conformal map) of the rays in a Euclidean geometry. To see this, let $\tilde{h}$ be an analytic function whose imaginary part is h, and let g be a map whose derivative is $e^{\tilde{h}}$.

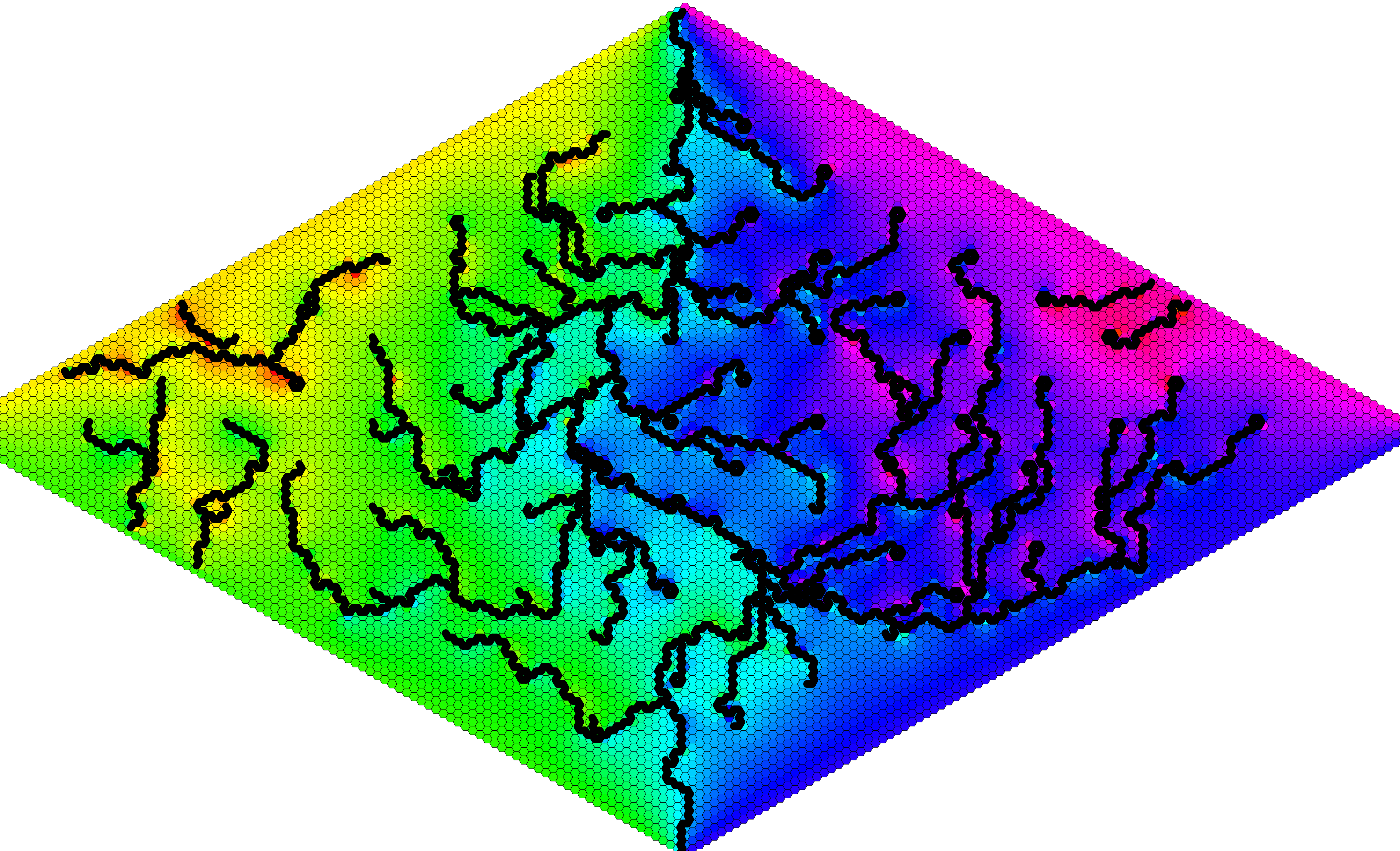
AC geometry of the GFF

Question: Is there a natural way to define the set of "flow lines" of $e^{ih/\chi}$ when χ is a constant and h is the continuous Gaussian free field? **Answer:** Yes. The flow lines are forms of SLE_κ where $0 < \kappa < 4$ and $\chi = \frac{4-\kappa}{2\pi}\lambda$. As in the case of contour lines, there is a constant "height gap" between one side of the flow line and the other. We may view this gap as an "angle gap." In radians, the gap is $\frac{\kappa\pi}{4-\kappa}$, i.e., $\frac{\kappa}{2(4-\kappa)}$ revolutions.

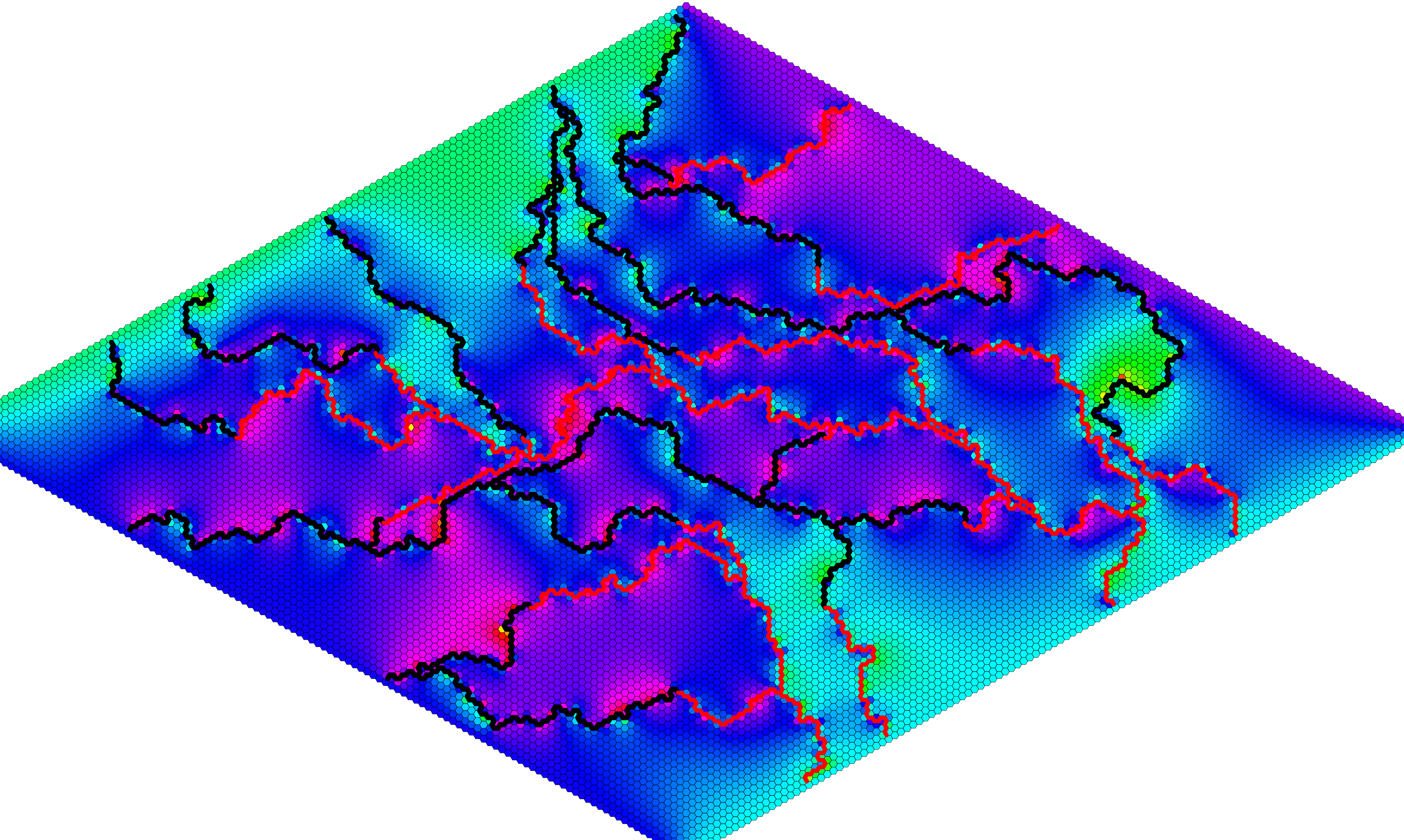
Discretized north-going line, $\kappa = .7$



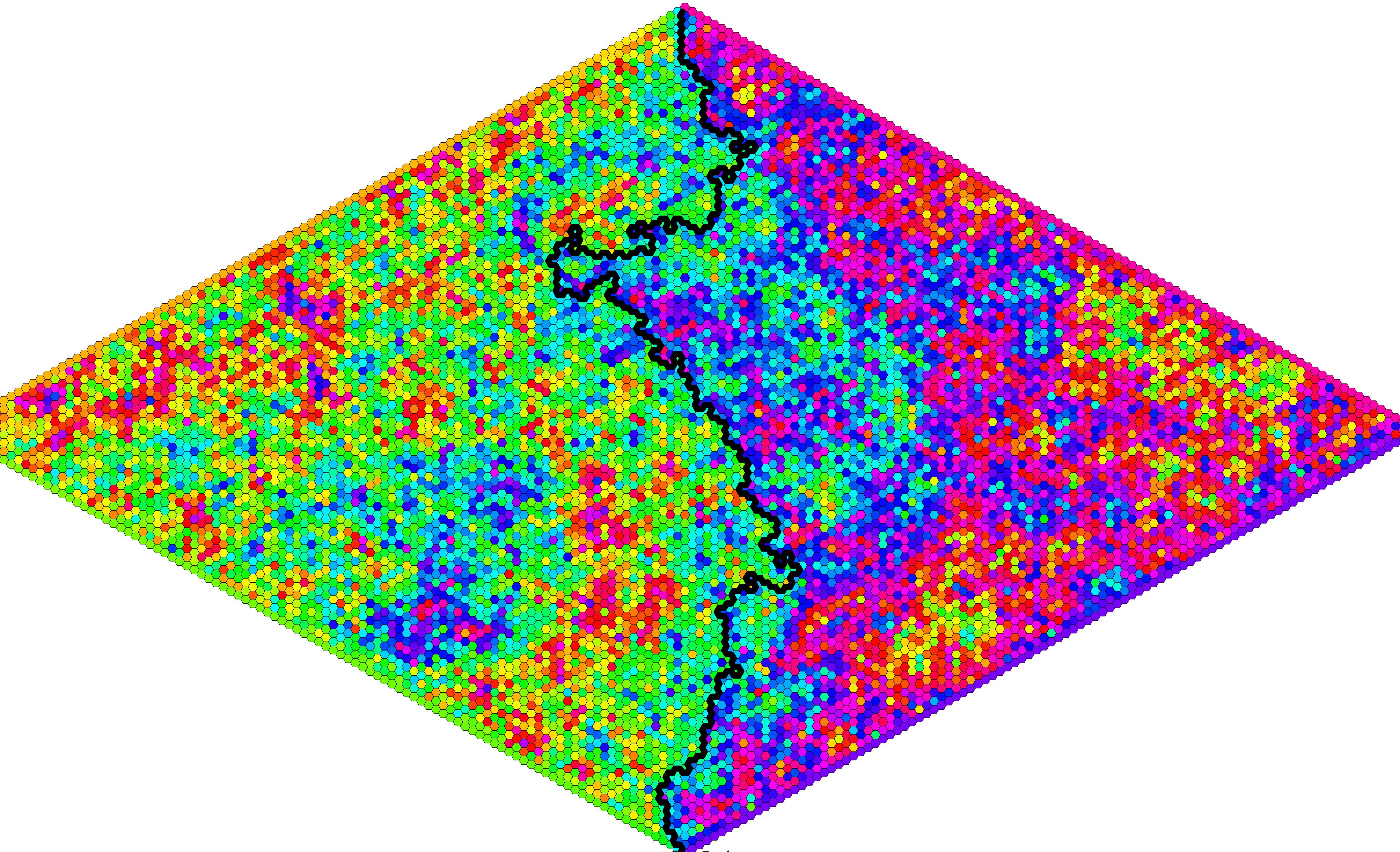
Discretized north-going tree, $\kappa = .7$



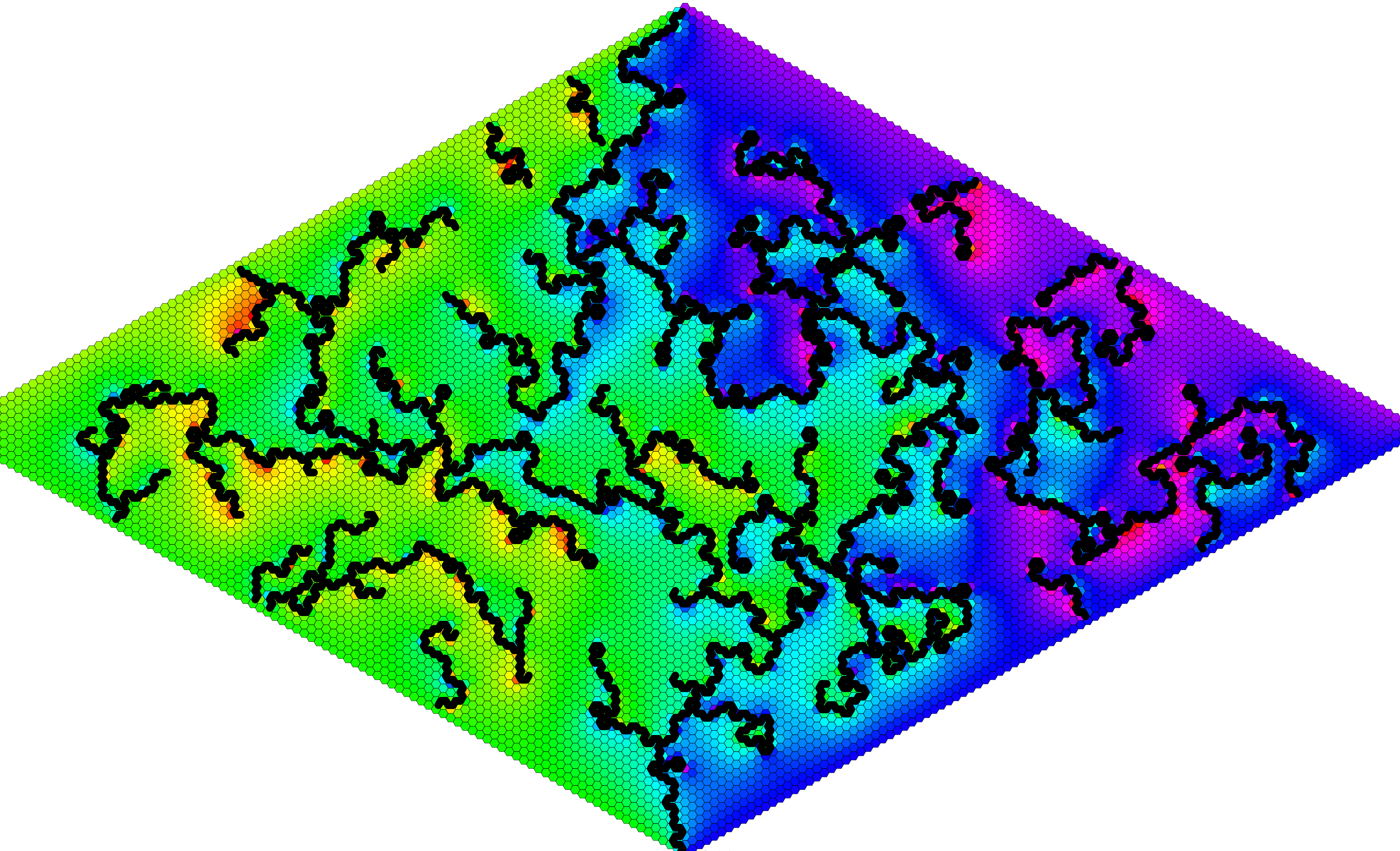
East/west-going trees, $\kappa = .7$



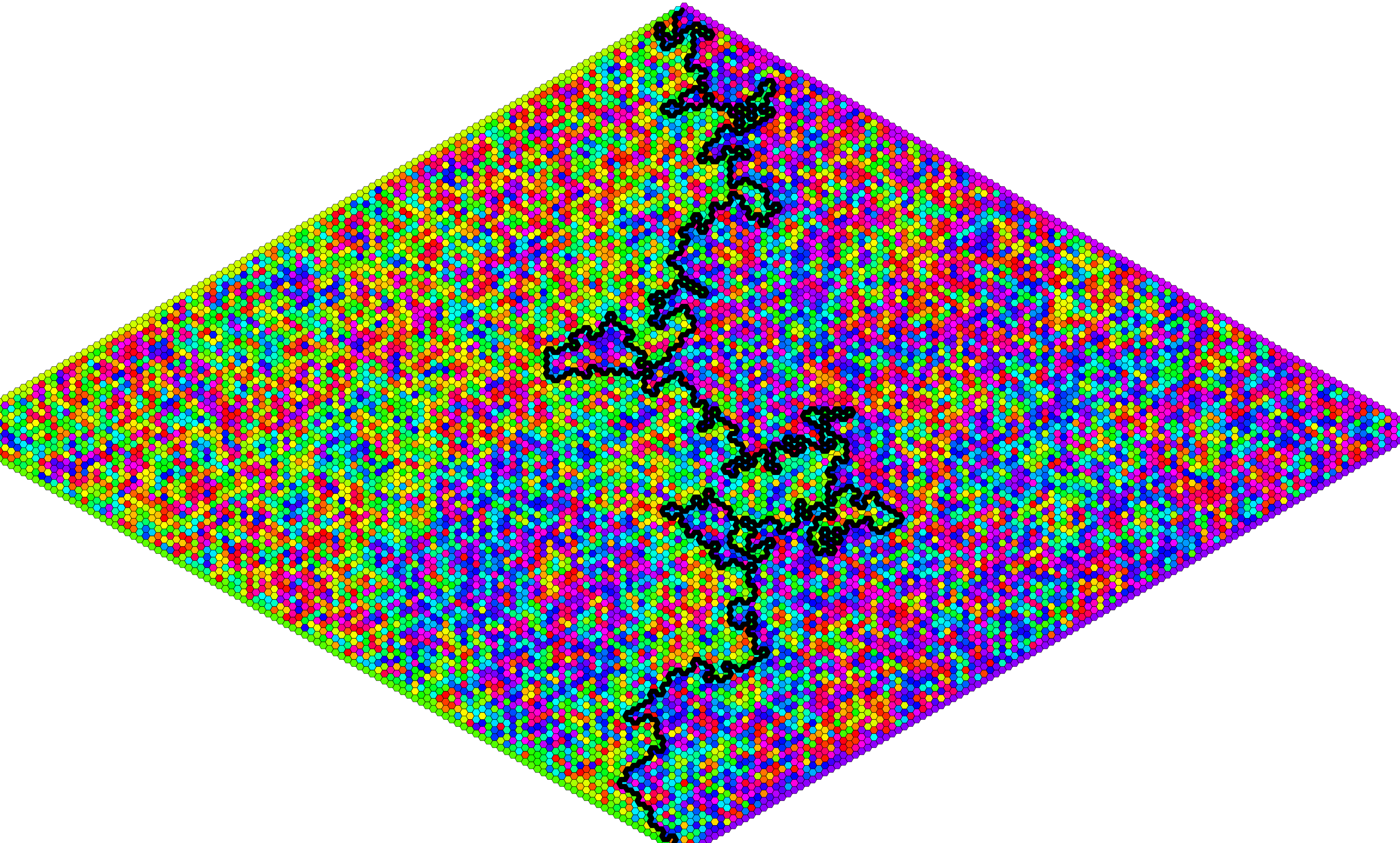
North-going line, $\kappa = 2$



North-going tree, $\kappa = 2$



North-going line, $\kappa = 3.5$



North-going tree, $\kappa = 3.5$