Mathematics of Finance
Midterm Preparation
October 12

To be thoroughly prepared for this midterm, you should

1. know how to do the homework problems — except the card problem and the proof (Problem 7) in problem set one, and the martingale problems in problem set three; these types of problems will not be on the exam.

2. be able to provide (correct!) definitions for the following words and understand what they mean: variance, expectation, covariance, correlation coefficient, random variable, call option, put option, stock, bond, annuity, perpetuity, martingale, risk-neutral probability, normal random variable, independent, conditional probability, arbitrage, replicating portfolio, present value, binomial tree model, short selling, solvency assumption, divisibility assumption, no-arbitrage principle, central limit theorem, optional stopping time theorem, fundamental theorem of asset pricing.

Below are some examples of problems that could be on the midterm:

1. Consider a three-sided die that comes up 4 with probability 1/2, 6 with probability 1/4, and 8 with probability 1/4.

1. Suppose we roll the die once and let Y be the value that comes up. Compute the expectation and variance and standard deviation of Y.

2. Suppose we roll the die 1000 times. Let X_i be the value that comes up on the ith roll. Let $X = \sum_{i=1}^{1000} X_i$ be the sum of the values that come up. Compute the expectation, variance, and standard deviation of Y.

3. What does the central limit theorem tell you about the probability that Y lies within one standard deviation of its expectation?

2. Suppose that there are three differently-shaped potatoes in a bag. One has length 6 inches and weight 2 ounces. One has length 6 inches and weight 5 ounces. One has length 3 inches and weight 2 ounces. Suppose

you draw a potato from the bag at random, so that each potato comes out with probability 1/3. Let L be the length of the potato you pick and W the weight.

1. Compute the expectation of L.
2. Compute the covariance of L and W.
3. Compute the variance of L
4. Compute the variance of W
5. Compute the correlation coefficient of L and W.
6. Compute the $\mathrm{Cov}(W + L + 32, 2W + L + 7)$.
7. Compute the conditional probability that the potato weighs two ounces given that it is six inches long.
8. Compute the conditional probability that the potato is six inches long given that it weights 2 ounces.
9. Are L and W independent? Why or why not?

4. Two ordinary six-sided dice are rolled. Let X be the sum of the values of the two dice. Compute the mean and variance of X.

5. Toss a fair coin a million times. What is the standard deviation of the number of heads you see?

6. Suppose that $S(0) = 1$ and that at each time step the value of $S(0)$ either goes up 100 percent or down 50 percent (each with probability 1/2) and that whether it goes up or down at one step is independent of what it does at any other step.

1. The expected value of $S(100)$.
2. Estimate the median value of $S(100)$.
3. Are these two values the same? Why or why not?

7. Suppose that $A(0) = 100$, $A(1) = 110$, $S(0) = 100$ and $S(1)$ is either equal to 115 or 85. Here A is price of riskless asset and S is the price of a stock. Let C be a call option with exercise time 1 and strike price 95.

1. Compute the risk neutral probability that $S(1) = 85$.
2. Describe a portfolio consisting of combinations of bonds and stocks whose value at time 1 is equal to the value of C in both scenarios (i.e., both when stock is up and when stock is down).
3. Compute $C(0)$ (assuming no arbitrage opportunities exist).
4. What is the expected value of $S(1)$ computed using the risk neutral probability?
5. What is the expected value of $S(1)/A(1)$, computed using the risk neutral probability?
6. Explain the following statement: $S(n)/A(n)$ and $C(n)/A(n)$ are both martingales with respect to the risk neutral probability.

8. What is meant by the phrase "(European) put option for Microsoft stock with exercise date September 31, 2006 and strike price 60"? What will the value of the option be on the exercise date if the price of Microsoft stock on the exercise date is 70?

9. Compute the present value of an annuity that pays 5 every year for ten years, starting exactly one year from now. Assume that the interest rate is .05.

10. Suppose that $A(0) = 100$, $A(1) = 110$, $S(0) = 50$, and $S(1)$ is either equal to 56 or 59, each with probability 1/2. Is there an opportunity for arbitrage? If not, why not? If so, give a portfolio with initial value zero whose value at time one is positive with probability one.

11. Define these words, in about two or three sentences each.

1. stock
2. annuity
3. divisibility assumption

4. binomial tree model

12. Suppose that $S(0) = 1$ and $S(1)$ is 2 with probability $1/2$ and .6 with probability $1/2$. Compute the *risk* (i.e., the standard deviation of the return) of owning stock during the time period between time 0 and 1.

13. Suppose that the bond always takes values $A(0) = 100$, $A(1) = 105$, $A(2) = 110$ but that the stock can take any of the following three possible sequences:

1. $S(0) = 50$, $S(1) = 40$, $S(2) = 20$
2. $S(0) = 50$, $S(1) = 40$, $S(2) = 60$
3. $S(0) = 50$, $S(1) = 60$, $S(2) = 80$
4. $S(0) = 50$, $S(1) = 60$, $S(2) = 50$

Then do the following

1. Draw a tree giving the possible values of S at times 0, 1, and 2.
2. At each of the three nodes of the tree where branching occurs (one at time 0, two at time 1), compute the risk neutral probability that the stock goes up.
3. Explain what is meant by the statement that $S(n)/A(n)$ is a martingale with respect to these probabilities.