Mathematics of Finance Midterm
October 12
100 points, 100 minutes

1. (20 points) Define these terms in about two or three sentences each. Be as complete as possible (without saying anything that is technically incorrect).

(a) short selling

(b) put option (with a given strike price and exercise date)

(c) optional stopping time theorem

(d) arbitrage

2. (25 points) Consider an ordinary six-sided die which assumes each of the values between one and six with equal probability. Roll the die and let E be equal to one if the roll is even and zero otherwise. Let L be equal to one if the roll is a "low roll," i.e., equal to 1, 2, or 3, and zero otherwise.

(a) Compute the expectation of E and the expectation of L.

(b) Compute the variance of L.

(c) Compute the covariance of E and L.

(d) Compute the covariance of $3E + L$ and $2L + 7$.

(e) Compute the conditional probability that $E = 1$ given that $L = 1$.

3. (15 points) Consider a coin that comes up heads with probability 1/3 and tails with probability 2/3. Suppose we toss the coin 450 times. Let H be the total number of heads. Let T be the total number of tails.

(a) Compute the standard deviation of H.

(b) Use the central limit theorem to approximate the probability that the H is between 145 and 175. (For example, you might say that this probability is approximately $\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ for some well-chosen values of a and b. You do not need to evaluate this integral explicitly.)

(c) Compute the covariance of H and L.

4. (20 points) Suppose that $A(0) = 100$, $A(1) = 110$, $S(0) = 10$ and $S(1)$ is either equal to 12 or 9. Here A is price of a riskless asset and S is the price of a stock. Let C be a call option for the stock with exercise time 1 and strike price 10.

(a) Compute the risk neutral probability that $S(1) = 12$.

(b) Describe a portfolio consisting of a combination of bond and stock whose value at time 1 is equal to the value of C in both scenarios (i.e., both when stock is up and when stock is down).

(c) Compute $C(0)$ (assuming no arbitrage opportunities exist).

5. (10 points) Suppose that $S(0) = 1$ and that at each time step the value of S either goes up 25 percent or down 20 percent (each with probability 1/2) and that whether it goes up or down at one step is independent of what it does at any other step.

(a) Compute the expected value of $S(50)$.

(b) Compute the median of $S(50)$ (i.e., the value a such that the probability that $S(50) \leq a$ is at least 1/2 and the probability that $S(50) \geq a$ is at least 1/2).

6. (10 points) Suppose that $A(0) = 20$, $A(1) = 22$, $S(0) = 50$, and that $S(1)$ can assume only the two values 46 and 54. Is there an opportunity for arbitrage? If so, give a portfolio with initial value zero whose value at time one is always positive.