Mathematics of Finance
Problem Set 9
Due December 7

I. We will now take a closer look at the Black-Scholes formulas for European calls and puts to see what we can learn from them.

(a) Look at the Black-Scholes formula for European calls (Theorem 8.6 of Capiínski and Zawstaniak). When parameters $t, S(t), X, T, r$ are fixed, what is the limit of $C^E(t)$ as σ tends to zero? (You may have to separately work out the two cases $X \geq e^{rT}$ and $X \leq e^{rT}$.) What is the formula for $C^E(t)$ when $\sigma^2 = 0$? (The Black-Scholes formula does not technically make sense in this case, since it involves a division by zero.) Explain your answer.

(b) Compute the values of delta, gamma, theta, vega, and rho (as defined in Section 9.1.2 of Zastawniak and Capiński) for a European call (showing your work) and check that you get the same answers that the text gives. (You may need to use the fact that $\frac{\partial}{\partial x}N(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.)

(c) Complete Exercise 9.6 (same problem but for European puts).

(d) You have now computed ten Greek parameters (five for puts, five for calls). For each of the these ten, determine whether the parameter is always positive, always negative, or sometimes positive and sometimes negative. (You may assume that T and σ are non-zero and $r > 0$.) In each case, give a rough intuitive explanation for why that is the case. Sample answer:

> "It is always the case that theta is less than zero for a call option. This is because the price of a call reflects the hope that the underlying asset will go up by a sizable amount by the expiry date; when time goes by without the stock going up at all, this is disappointing to holders of call options (since it decreases the likelihood of big gains for the holder of the call), and so the price of the call option goes down."

(e) Compute the second derivative of $C^E(t)$ with respect to X. What is the relationship between this function (as a function of X, with other parameters fixed) and the risk neutral probability?

II. We now explore an inefficient market scenario in which market timing is possible for certain skilled traders. Suppose that a particular stock had one million outstanding shares, valued at \$100 each at the beginning of 2004 and \$112 each at the end of 2004. For every investor in the stock, we can count the total number of "share-years" that that investor held during 2004; this is the sum over all the shares that the investor held during 2004 times the fraction of a year that the investor held that share. (For example, an investor who held 300 shares for one third of a year held "100 share years" during 2004. An investor who held 100 shares for the entire year also held 100 share-years during 2004.) Now, suppose that we can divide these share-years into three types of investors:

1. Long-term investors, who held their shares the entire year, held 80 percent of the shares (hence 80 percent of the share-years).
2. Skilled market timers (insiders, hedge fund analysts, etc.) held 5 percent of the share years. By buying low and selling high, the investors managed to earn on average 24 dollars per share-year.
3. Ordinary short term investors held 15 percent of the share-years; these people held some shares for less than the entire year but did not have any special market timing skills.

Now answer the following:

(a) How much did the long-term investors earn, on average, per share-year in 2004?

(b) How much did the ordinary short term investors earn, on average, per share-year in 2004?

(c) Suppose that we modify the problem to include short selling, and there were 100,000 share-years sold short during 2004, and hence a total of 1.1 million long share-years. Suppose that 150,000 of the long share-years belong to successful market timers, who earned on average \$24 per share-year on these shares. Suppose that the short sellers were also successful market times and managed to earn only 6 percent a year on average. (Short sellers seek to minimize the percentage they earn, since they hold a negative amount of the stock.) Again, let us suppose that 800,000 share years belonged to long-term investors and 150,000 to ordinary short-term investors. How much did the ordinary short-term investors earn per share-year, on average, in this scenario?

III. Binomial tree model design: take 252 to be the number of trading days in the year, σ annual the historical volatility and r_{ANNUAL} the annual interest rate and g the expected annual return on the risky asset. Show that a binomial tree model with time step equal to one trading day—taking $r = r_{ANNUAL}/252$, $u = g/252 + \sigma/\sqrt{252}$, $d = g/252 - \sigma/\sqrt{252}$, $p = 1/2$—gives annual volatility of approximately σ and an expected return on the risky asset of approximately g.

IV. Now consider the binomial tree model, with each unit of time representing one trading day. Take $g = .11$ and $r_{ANNUAL} = .05$ and $\sigma = .2$ and use the model as given in the previous problem (i.e., $r = .05/252$, $u = .11/252 + .2/\sqrt{252}$, $d = .11/252 - .2/\sqrt{252}$). (We can imagine the risky asset to be an NYSE index fund and the safe asset to be a bond. In this case, the annual expected returns and the volatility in our model are not too far from the historical averages.)

(a) Suppose an investor has "utility function" $U(x) = \log(x)$ and the investor's wealth at time n is denoted $W(n)$. Given the investor's wealth $W(0)$, how much money should the investor put in the stock and how much in the bond in order to maximize the expectation of $U(W(1))$?

(b) If you did the previous part correctly, you found that the fraction of wealth that the investor should put in the stock is independent of $W(0)$ and (moreover) is greater than one (i.e., the investor is borrowing money to invest in the stock market). If we assume that the investor is only able to borrow money at an annual rate of about 8 percent (so $r = .08/252$), how does this change the answer to the previous question?