Mathematics of Finance
Problem Set 8
Due November 30

I. Complete Exercises 8.7, 8.8, 8.9, 8.12, 8.13, 8.14, and 8.15 of Zawstaniak and Capiński.

II. In the binomial tree model, assume $A(0) = S(0) = 1$, $r = .03$, $u = .15$ and $d = -.05$. The probability that the stock goes up at each step is $p = .5$.

(a) Calculate the risk neutral probability p_* that the stock goes up at a given step.

(b) Graph the probability distribution for $\log S(5)$. That is, write down the six possible values of $\log S(5)$, and draw a histogram showing the probability of each of them. Do the same using risk neutral probability.

(c) Draw an approximate sketch of the probability distribution for $\log S(1000)$ by using the central limit theorem and drawing a normal density function of the appropriate mean and variance. Do the same using risk neutral probability.

III. In the binomial tree model, assume $A(0) = S(0) = 1$, and for some fixed value of a, b, c and some $N > 0$ we have $r = a/N^2$, $u = b/N^2 + c/N$ and $d = b/N^2 - c/N$. We suppose that N^2 units of time correspond to one year — that is, we are dividing a year into N^2 discrete time intervals. For example, if $N = 16$, then $N^2 = 256$, which is approximately the number of trading days in a year. So we could think of each time increment as representing one trading day. Alternatively, we could take $N = 160$ and $N^2 = 25600$, and let each increment represent a hundredth of a trading day. Let $p = 1/2$ be the probability that stock goes up during a given time increment.

(a) Compute the expectation and variance of $\log S(N^2)$ (in terms of a, b, c, and N). Compute the limit of these values when N tends to infinity.

(b) Compute the expectation and variance of $\log S(N^2)$ with respect to the risk neutral probability. Compute the limits of these values as N tends to infinity.

IV. The purpose of this problem is to derive the Black-Scholes formula for a call option with strike price X and exercise time T and underlying volatility σ using only the assumption that the risk neutral probability is lognormal and without mentioning Brownian motion (as Capiński and Zastawniak do). We take the current time to be $t = 0$. Suppose that Y is a normal random variable with expectation mT and variance $\sigma^2 T$.

(a) Compute the expectation of e^Y.

(b) If e^Y is the risk neutral probability distribution of $S(T)$, then the martingale property implies that the risk neutral expectation of e^Y must be $S(0)e^{rT}$. Using this fact and the answer to (a), compute m as a function of r, T, $S(0)$ and σ.

(c) Fix some $X > 0$ and let $f(x)$ be the function which is 0 when $x \leq X$ and $x - X$ when $x > X$. Compute the expectation of $f(e^Y)$ in terms of r, X, T, σ, and $S(0)$. Interpret this value as the price of a call option and check that your answer agrees with the Black-Scholes formula (Theorem 8.6 of Capi'nski and Zastawniak).

V. Go to, e.g., www.cboe.com, and look up the call options prices for Google for January, 2007 (use the average of the bid and ask price). Compute the implied volatilities corresponding to the following strike prices: $X = 200, 400, 570$, assuming no dividends and a risk-free interest rate of $r = 4.5$ percent (and approximating T by 1.2 years). That is, for each such X, find what σ would have to be in order for the Black-Scholes formula to give the average between the bid and ask price. (If necessary, you may use trial and error to find a σ that is approximately correct.) Compare this to implied volatility and historical volatility numbers that you find quoted at www.iseoptions.com. (Type in GOOG into the "Most Active Options Classes" window at this website and when a chart comes up, click on "GOOG," so that it brings up some options prices and historical and implied volatility numbers.)

VI. Suppose that the last 10 closing values for a stock were (from oldest to most recent) $100, 101, 100, 99, 102, 105, 103, 106, 102, 104$. Compute the nine-day historical volatility.